

EXISTENCE AND NO EXISTENCE OF POSITIVE SOLUTIONS FOR ROBIN (P,Q)-LAPLACE EQUATION WITH TWO PARAMETERS

Selma Hadjer Djefal

post graduate student, Laboratory of applied Mathematics, Department of mathematics, Badji Mokhtar university Algeria

sel.djefal@gmail.com

ORCID: <https://orcid.org/0000-0002-1114-4267>.

Lamine Bouzettouta

M.C.A department of mathematics Skikda university, Algeria

Lami_750000@yahoo.fr

Zoubida Marsa, MCA

Laboratory of Chemical and Environmental Engineering Skikda, Algeria

z.marsa@univ-skikda.dz

Received: 02/02/2025

Accepted: 05/05/2025

Published: 09/07/2025

Abstract:

The purpose of this paper is to study the (p,q)-Laplace equation under the Robin boundary condition in a bounded region of \mathbb{R}^N through the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \gamma |u|^{p-2} u + \mu |u|^{q-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta_{pq}} + \beta_1 |u|^{p-2} u + \beta_2 |u|^{q-2} u & \text{on } \partial\Omega \end{cases}$$

The principal results of this article concerns the existence and not existence of positive solution. For this we will discuss all the possible cases depending on the parameters γ, μ and the first eigenvalue of p-Laplacian (respectively q-Laplacian). Using various amount of methods mostly the Min-Max theory wich include the Palais-Smale condition, the global and local minimizer and the Col theorem, We will also use the notion of super solution. Adding to that we will

give a definition for the curve witch separates between the regions of existence and not existence of positive solution in (γ, μ) plane and we will make sur that this curve wich we will be noted C is continuous and locally finite. Moreover we are going to explor the existence of positive solution in the region where the curve C is included. We have chosen to put our attention on the study of (p, q) -Laplace equation under the Robin boundary condition because of its importance for modeling nonlinear phenomena in physics, biology, ecology, and other fields. We can take as an example in ecology it can be used to modeling the phenomena of population dynamics or in physics to modeling the nonlinear heat conduction.

Keywords in English:

positive solution, Robin boundary condition, eigenvalue, (p, q) -Laplacian, Min-Max theory.

2000MSC: 35J20, 35J65, 35P30.

1. Introduction

In this article, we are interested with non-existence and existence of positive solutions for the following problem involving the (p, q) -Laplacian operator

$$\begin{cases} -\Delta_p u - \Delta_q u = \gamma |u|^{p-2} u + \mu |u|^{q-2} u & \text{in } \Omega'' \\ \frac{\partial u}{\partial \eta_{pq}} + \beta_1 |u|^{p-2} u + \beta_2 |u|^{q-2} u & \text{on } \partial\Omega' \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded region with smooth boundary $\partial\Omega$ and γ, μ are two real parameters, $\beta_1, \beta_2 > 0$ and $1 < p < q < \infty$.

We define the operator Δ_m by the usual m -Laplacian such that

$$\Delta_m u := \operatorname{div}(|\nabla u|^{m-2} \nabla u),$$

with $m \in (1, +\infty)$. $W^{1,m}(\Omega)$ denote the standard Sobolev space equipped with the quantity

$$\|u\|_{1,m} := \left(\|u\|_{m,\Omega}^m + \|u\|_{m,\partial\Omega}^m \right)^{\frac{1}{m}},$$

where $\|u\|_{m,\Omega} := \left(\int_{\Omega} |u|^m dx \right)^{\frac{1}{m}}$ and $\|u\|_{m,\partial\Omega} := \left(\int_{\partial\Omega} |u|^m d\sigma \right)^{\frac{1}{m}}$, with σ is the $(N-1)$ -Hausdorff surface measure of \mathbb{R}^{N-1} .

We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1) if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla v dx + \beta_1 \int_{\partial\Omega} |u|^{p-2} u v d\sigma + \beta_2 \int_{\partial\Omega} |u|^{q-2} u v d\sigma$$

$$= \gamma \int_{\Omega} |u|^{p-2} u v dx + \mu \int_{\Omega} |u|^{q-2} u v dx, \forall v \in W^{1,p}(\Omega) \tag{2}$$

Then the functional associated with problem (1) is given by

$$I_{\gamma,\mu}(u) := \frac{1}{p} \|\nabla u\|_{p,\Omega}^p + \frac{1}{q} \|\nabla u\|_{q,\Omega}^q + \frac{\beta_1}{p} \|u\|_{p,\partial\Omega}^p + \frac{\beta_2}{q} \|u\|_{q,\partial\Omega}^q - \frac{\gamma}{p} \|u_+\|_{p,\Omega}^p - \frac{\mu}{q} \|u_+\|_{q,\Omega}^q .$$

The interest in studying nonlinear partial differential equations with the (p,q)-Laplacian operator has increased in the last decade because numerous problems arising in plasma physics [1], reaction-diffusion equations ([2],[3]), biophysics [4] and elementary particles ([5], [6], [7], [8]) may be stated in this form.

The study of equation (1) begin where Motreanu and all in [9] introduced the following eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda(m_p(x) |u|^{p-2} u + m_q(x) |u|^{q-2} u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} ,$$

Where weights $m_p, m_q \in L^\infty(\Omega)$

In [10] , provides five typical examples of the bifurcation of positive solutions for the one-dimensional (p,q)-Laplace equation on the interval (-L,L) by using the time maps

$$\begin{cases} (|u'|^{p-2} u)' + (|u'|^{q-2} u)' + \lambda(|u|^{p-2} u + |u|^{q-2} u) = 0 \\ u(-L) = u(L) = 0 \end{cases} ,$$

They proved that the bifurcation curve change depending on p,q and L.

All the results of this paper are valid for the problem (1) with non-negative weights

$$\begin{cases} -\Delta_p u - \Delta_q u = \gamma P(x) |u|^{p-2} u + \mu Q(x) |u|^{q-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta_{pq}} + \beta_1 |u|^{p-2} u + \beta_2 |u|^{q-2} u & \text{on } \partial\Omega \end{cases} ,$$

where P,Q are positive weights $P, Q \in L^\infty(\Omega)$.

In this paper we employ an appropriate min-max approach and super and sub-solution method to show the existence and non-existence of positive solutions for (1) problem, and construction of continuous curve in (γ, μ) .

We say that λ is an eigenvalue of $-\Delta_m$ with weight $P_m \in L^\infty(\Omega)$ if the problem

$$\begin{cases} -\Delta_m u = \lambda P_m(x) |u|^{m-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} + \beta |u|^{m-2} u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

possesses a non-trivial solution.

If the Lebesgue mesure of $\{x \in \Omega, P_m(x) > 0\}$ is positive, we can characterize the first positive eigenvalue

$$\lambda_1(m, P_m) := \inf \left\{ \frac{\int_{\Omega} |\nabla u|^m dx}{\int_{\Omega} P_m |u|^m dx} + \beta \int_{\partial\Omega} |u|^m d\sigma : u \in W^{1,m}(\Omega), \int_{\Omega} P_m |u|^m dx > 0 \right\},$$

λ is simple and isolated and the corresponding eigenfunction $\psi_1(m, P_m) \in C^{1,\alpha_m}(\overline{\Omega})$; $\alpha_m \in (0,1)$. We will use the notation $\lambda_1(m)$ for Δ_m with $P_m \equiv 1$.

We remind that a function $u \in W^{1,p}(\Omega)$ is called a super-solution (respectively a sub-solution) of the problem (1) if $u \geq 0$ (respectively $u \leq 0$) on $\partial\Omega$ and for any $\psi \in W^{1,p}(\Omega)$ satisfying $\psi(x) \geq 0$ a.e. $x \in \Omega$ we have

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \psi dx - \beta_1 \int_{\partial\Omega} |u|^{p-2} u \psi d\sigma - \beta_2 \int_{\partial\Omega} |u|^{q-2} u \psi d\sigma \\ & - \gamma \int_{\Omega} |u|^{p-2} u \psi dx - \mu \int_{\Omega} |u|^{q-2} u \psi dx \geq 0, \text{ (respectively } \leq 0) \end{aligned}$$

For more details, see [11].

In what follows, we will stipulate that $\lambda_1(p)$ and $\lambda_1(q)$ possesses a different eigenspaces if the corresponding ψ_p and ψ_q are linearly independent which means that the following supposition is verified

$$\forall t \neq 0, \text{ it holds } \psi_p \neq t\psi_q, \text{ in } \Omega \quad (4)$$

2. Results and Discussion

the main results of our paper and their In this section proofs are given.

We start by the following theorem.

Theorem 1 : If $(\gamma, \mu) \in (-\infty, \lambda_1(p)] \times (-\infty, \lambda_1(q)] \setminus \{\lambda_1(p), \lambda_1(q)\}$ holds, then the problem (1) has no a non trivial solution. Moreover, for $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$ (1) has a non-trivial solution if and only if $\lambda_1(p)$ and $\lambda_1(q)$ have the same eigenspace which means that (4) is not satisfy.

Proof of Theorem 1 : Let $\gamma < \lambda_1(p)$ and $\mu < \lambda_1(q)$. Suppose that $u \in W^{1,p}(\Omega)$ is a non-trivial solution of (1). Taking $v = u$ into relation (2) and by Poincare inequality we get

$$0 \leq (\lambda_1(p) - \gamma) \|u\|_{1,p}^p \leq \|\nabla u\|_{p,\Omega}^p - \gamma \|u\|_{p,\Omega}^p = \mu \|u\|_{q,\Omega}^q - \|\nabla u\|_{q,\Omega}^q \leq (\mu - \lambda_1(q)) \|u\|_{1,q}^q \leq 0.$$

Previous inequalities are satisfied if and only if $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$ which is a contradiction.

We put now $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$. If $u \in W^{1,p}(\Omega)$ is a solution of (1) then we have $0 \leq \|\nabla u\|_{p,\Omega}^p - \lambda_1(p) \|u\|_{p,\Omega}^p = \lambda_1(q) \|u\|_{q,\Omega}^q - \|\nabla u\|_{q,\Omega}^q \leq 0$ which give that (4) is not verified.

Suppose that ψ_p and ψ_q have the same eigenspaces i.e (4) is violated.

Hence, the functions ψ_p and ψ_q are both non-trivial solutions of (1) with $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$ and this finishes the proof. ■

Theorem 2 : If we have $(\lambda, \mu) \in (\lambda_1(p), +\infty) \times (-\infty, \lambda_1(q)) \cup (-\infty, \lambda_1(p)) \times (\lambda_1(q), +\infty)$ then (1) has at least one positive solution.

Following the same arguments of theorem 1.2 in [9] we get the next Lemma

Lemma 1 : Let $P, Q \in L^\infty(\Omega)$ positive weights.

Assume that $(\lambda_1(p), P) \neq (\lambda_1(q), Q)$, hold. If λ is such that

$$\min\{(\lambda_1(p), P(x)), (\lambda_1(q), Q(x))\} < \lambda < \max\{(\lambda_1(p), P(x)), (\lambda_1(q), Q(x))\}$$

then the following problem has at least one positive solution.

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda(P(x) |u|^{p-2} u + Q(x) |u|^{q-2} u) & \text{in } \Omega \\ \frac{\partial u}{\partial \eta_{pq}} + \beta_1 |u|^{p-2} u + \beta_2 |u|^{q-2} u & \text{on } \partial\Omega \end{cases} \tag{5}$$

Proof of Theorem 2: We split the proof into several cases:

Case I: If we have $\gamma > \lambda_1(p)$ and $0 < \mu < \lambda_1(q)$ or $0 < \gamma < \lambda_1(p)$ and $\mu > \lambda_1(q)$

For the first situation, it yields that

$$\lambda_1\left(p, \frac{\gamma}{\mu}\right) = \frac{\lambda_1(p)\mu}{\gamma} < \mu < \lambda_1(q) = \lambda_1(q, 1).$$

Now, setting $\lambda = \mu$, $P = \frac{\gamma}{\mu}$ and $Q = 1$ in lemma 1 we get the result.

The same argument show the result in the second situation with $\lambda = \gamma$, $Q = \frac{\mu}{\gamma}$ and

$$P = 1.$$

Case II : Let $\gamma > \lambda_1(p)$ and $\mu \leq 0$.

We introduce an alternative functional $I_{0,\gamma,\mu}$ defined on $W^{1,p}(\Omega)$ by

$$I_{0,\gamma,\mu}(u) := \frac{1}{p} \|\nabla u\|_{1,p}^p + \frac{1}{q} \|\nabla u\|_{1,q}^q - \frac{\gamma}{p} \|u_+\|_{p,\Omega}^p - \frac{\mu}{q} \|u_+\|_{q,\Omega}^q$$

One we easily show that the functional $I_{0,\gamma,\mu}$ satisfies the Palais-Smale condition.

Moreover, for the functional $I_{0,\gamma,\mu} + \frac{\mu}{q} \|u_+\|_{q,\Omega}^q$ there exist $\delta > 0$ and $\rho > 0$ such

that $I_{0,\gamma,\mu} + \frac{\mu}{q} \|u_+\|_{q,\Omega}^q \geq \delta$ provided $\|u\|_{p,\Omega} = \rho$.

Since $\mu \leq 0$ then we get $I_{0,\gamma,\mu} \geq \delta$ for $\|u\|_{p,\Omega} = \rho$.

For $k > 0$ sufficiently large and ψ_p a positive eigenfunction corresponding to $\lambda_1(p)$ we have

$$I_{0,\gamma,\mu}(k\psi_p) = \frac{k^p}{p} (\lambda_1(p) - \gamma) \|\psi_p\|_{p,\Omega}^p + \frac{k^q}{q} (\|\nabla \psi_p\|_{q,\Omega}^q - \mu \|\psi_p\|_{q,\Omega}^q) < 0.$$

The existence result in this case follows immediately by applying Mountain Pass theorem.

Case III : Suppose now that $\gamma \leq 0$ and $\mu > \lambda_1(q)$.

It easy to proof that $I_{0,\gamma,\mu}$ is bounded from below and coercive due to the fact $q < p$.

Moreover, by the compactness embedding of the space $W^{1,p}(\Omega)$ into $L^p(\Omega)$ and $L^q(\Omega)$ respectively $I_{0,\gamma,\mu}$ is weakly lower semi continuous then, it possess a global minimizer $u \in W^{1,p}(\Omega)$.

On the other hand, for $k > 0$ sufficiently small and ψ_q the positive eigenfunction corresponding to $\lambda_1(q)$, we have

$$I_{0,\gamma,\mu}(k\psi_q) = \frac{k^q}{q} (\lambda_1(q) - \mu) \|\psi_q\|_{q,\Omega}^q + \frac{k^p}{p} (\|\nabla \psi_q\|_{p,\Omega}^p - \gamma \|\psi_q\|_{p,\Omega}^p) < 0.$$

Then $\min_{u \in W^{1,p}(\Omega)} I_{0,\gamma,\mu}(u) < 0$ and u is a non trivial solution of (1). ■

Now, we have the problem (1) is equivalent to the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u = (\lambda + s) |u|^{p-2} u + \lambda |u|^{q-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta_{pq}} + \beta_1 |u|^{p-2} u + \beta_2 |u|^{q-2} u & \text{on } \partial\Omega \end{cases} \tag{6}$$

Where $s = \gamma - \lambda$. To make it easier we put $\gamma = \lambda + s$ and $\mu = \lambda$, for each $s \in \mathbb{R}$ we consider,

$$\lambda^*(s) := \sup \{ \lambda \in \mathbb{R}; (6) \text{ has a positive solution} \} \tag{7}$$

In the case where there is no λ we put $\lambda^*(s) = -\infty$. Moreover we define

$$s^* := \lambda_1(p) - \lambda_1(q).$$

In the following proposition we provide important properties of $\lambda^*(s)$.

Proposition 1 : Let $\lambda^*(s), s \in \mathbb{R}$ defined by (7), then we have

- 1- $\lambda^*(s) < +\infty, \forall s \in \mathbb{R}$,
- 2- $\lambda^*(s^*) + s^* > \lambda_1(p)$ and $\lambda^*(s^*) > \lambda_1(q)$ if and only if (4) is satisfied,

- 3- $\lambda^*(s) + s \geq \lambda_1(p)$ or $\lambda^*(s) \geq \lambda_1(q), \forall s \in \mathbb{R} - \{s^*\}$,
- 4- $\lambda^*(s)$ is non-increasing and $\lambda^*(s) + s$ is non decreasing on \mathbb{R} ,
- 5- $\lambda^*(s)$ is continuous on \mathbb{R} .

Proof of proposition 1 : We start by the first properties

1- Let $s \in \mathbb{R}$ - fixed and $u \in W^{1,p}(\Omega)$ a positive solution of (6) then

$u \in \text{int } C^1(\bar{\Omega})_+$ see [12], we can choose any $\psi \in \text{int } C^1(\bar{\Omega})_+$ and by taken

$v = \frac{\psi^p}{u^{p-1} + u^{q-1}}$ into relation (2) we have

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx \\ & + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma \\ & = \gamma \int_{\Omega} |u|^{p-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx + \mu \int_{\Omega} |u|^{q-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx \end{aligned}$$

by the properties of Picone identity of (p, q) -Laplacian there exist ρ such that

$$(|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \nabla \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) \leq \frac{|\nabla \psi^p| + |\nabla \psi^{p/q}|^q}{\rho}$$

Then we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) dx \\ & + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma \\ & = \lambda \int_{\Omega} \psi^p dx + s \int_{\Omega} \frac{u^{p-1} \psi^p}{u^{p-1} + u^{q-1}} dx + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma \\ & \leq \frac{|\nabla \psi^p| + |\nabla \psi^{p/q}|^q}{\rho} + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi^p}{u^{p-1} + u^{q-1}} \right) d\sigma \end{aligned}$$

then

$$\lambda \int_{\Omega} \psi^p dx + s \int_{\Omega} \frac{u^{p-1} \psi^p}{u^{p-1} + u^{q-1}} dx \leq \frac{|\nabla \psi^p| + |\nabla \psi^{p/q}|^q}{\rho}.$$

Combining the last inequality with the following estimation,

$$\lambda \int_{\Omega} \psi^p dx + \min \left\{ 0, s \int_{\Omega} \psi^p dx \right\} \leq \frac{1}{\rho} \left(\int_{\Omega} |\nabla \psi|^p dx + \int_{\Omega} |\nabla \psi^{p/q}|^q dx \right).$$

Since

$$\rho, \int_{\Omega} |\nabla \psi|^p \text{ and } \int_{\Omega} |\nabla \psi^{p/q}|^q,$$

Are finites and positives then

$\lambda^*(s) < +\infty, \forall s \in \mathbf{R}$.

2- Suppose that (4) is satisfied.

We introduce the Nahari Manifold defined by

$$N_{\gamma,\mu} := \left\{ u \in W^{1,p}(\Omega) \setminus \{0\}, \langle I'_{\gamma,\mu}(u), u \rangle = 0 \right\} .$$

It's easy to show that if $u \in N_{\gamma,\mu}$ is a critical point of $I'_{\gamma,\mu}$ on $N_{\gamma,\mu}$ then u is a critical point of $I'_{\gamma,\mu}$ on $W^{1,p}(\Omega)$. Moreover, we can easily prove that there exist $\varepsilon^* > 0$ such that

$$\begin{aligned} & \int_{\Omega} |\nabla u_{\varepsilon}|^q dx - \lambda_1(q) + \varepsilon \int_{\Omega} |u_{\varepsilon}|^q dx + \beta_2 \int_{\partial\Omega} |u_{\varepsilon}|^q dx < 0 \\ & < \int_{\Omega} |\nabla u_{\varepsilon}|^p dx - \lambda_1(p) + \varepsilon \int_{\Omega} |u_{\varepsilon}|^p dx + \beta_1 \int_{\partial\Omega} |u_{\varepsilon}|^p dx \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon^*)$. Where u_{ε} is the solution of problem (1) with $\gamma = \lambda_1(p) + \varepsilon$ and $\mu = \lambda_1(q) + \varepsilon$. We easily show that $u_{\varepsilon} \in N_{\gamma,\mu}$ with $\gamma = \lambda_1(p) + \varepsilon$ and $\mu = \lambda_1(q) + \varepsilon$ is a minimizer of

$I_{\lambda_1(p)+\varepsilon, \lambda_1(q)+\varepsilon}$ for all $\varepsilon \in (0, \varepsilon^*)$. Since $I_{\gamma,\mu}$ is even, we can conclude by the Min-Max theorem that the problem (1) with $\gamma = \lambda_1(p) + \varepsilon$ and $\mu = \lambda_1(q) + \varepsilon$ possess a positive solution for $\varepsilon \in (0, \varepsilon^*)$.

Since $(\lambda_1(p) + \varepsilon, \lambda_1(q) + \varepsilon) = (\lambda_1(q) + \varepsilon + s^*, \lambda_1(q) + \varepsilon)$ and by the definition of $\lambda^*(s^*)$ then we have $\lambda^*(s^*) \geq \lambda_1(q) + \varepsilon$ so $\lambda^*(s^*) + s^* \geq \lambda_1(q) + \varepsilon + s^*$ then $\lambda^*(s^*) + s^* \geq \lambda_1(p) + \varepsilon$. Assume now that (4) is not satisfied. Suppose that $u \in C^1(\bar{\Omega})_+$ is a positive weak solution of problem (Eq1) then by the classical Picone inequality [13] we have

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma \\ & \leq \int_{\Omega} |\nabla \psi_p|^p dx + \beta_1 \int_{\partial\Omega} |\psi_p|^p d\sigma = \lambda_1(p) \int_{\Omega} \psi_p^p dx . \end{aligned} \tag{8}$$

and by the generalized Picone's inequalities we have

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma \\ & \leq \int_{\Omega} |\nabla \psi_p|^{q-2} \nabla \psi_p \nabla \left(\frac{\psi_p^{p-q+1}}{u^{p-q}} \right) dx + \beta_2 \int_{\partial\Omega} |\psi_p|^{q-2} \psi_p \left(\frac{\psi_p^{p-q+1}}{u^{p-q}} \right) d\sigma \\ & = \lambda_1(q) \int_{\Omega} \psi_p^q u^{q-p} dx . \end{aligned} \tag{9}$$

Then we obtain the following inequality

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx$$

$$\begin{aligned}
 & +\beta_1 \int_{\partial\Omega} u^{p-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma + \beta_2 \int_{\partial\Omega} u^{q-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma \\
 & = \gamma \int_{\Omega} \psi_p^p dx + \mu \int_{\Omega} \psi_p^p u^{q-p} dx \leq \lambda_1(p) \int_{\Omega} \psi_p^p dx + \lambda_1(q) \int_{\Omega} \psi_p^p u^{q-p} dx
 \end{aligned}$$

Which is impossible in the case where $\gamma > \lambda_1(p)$ and $\mu > \lambda_1(q)$ simultaneously.

3- If $s \neq s^*$ and for $\gamma = \lambda + s$ and $\mu = \lambda$, by theorem 2 we have $\lambda^*(s) + s \geq \lambda_1(p)$ or $\lambda^*(s) \geq \lambda_1(q)$.

4- Let $s < s'$ we prove that $\lambda^*(s) \geq \lambda^*(s')$. By part **3**, we have $\lambda^*(s') \geq \lambda_1(q)$. Fix any $\varepsilon > 0$ such that $\lambda^*(s') - \varepsilon \geq \lambda_1(q)$ then by the definition of $\lambda^*(s')$ there exist λ such that $\lambda^*(s') - \varepsilon < \lambda < \lambda^*(s')$ then the problem (1) with $\gamma = \lambda + s'$ and $\mu = \lambda$ has a positive solution $u_\lambda \in C^1(\bar{\Omega})_+$. Since we have $s < s'$ then u_λ is a super solution of the problem (1) with $\gamma = \lambda + s'$ and $\mu = \lambda$ then there exist $w \in C^1(\bar{\Omega})_+$ solution of (1) with $\gamma = \lambda + s'$ and $\mu = \lambda$. As ε is arbitrary and $\lambda^*(s) \geq \lambda > \lambda^*(s') - \varepsilon$ then $\lambda^*(s) \geq \lambda^*(s')$.

Now, for $s < s^*$ we show that $\lambda^*(s) + s \leq \lambda^*(s') + s'$. If $\lambda^*(s) + s - s' \leq \lambda_1(q)$, as we have $\lambda_1(q) \leq \lambda^*(s')$ then $\lambda^*(s) + s \leq \lambda_1(q) + s' \leq \lambda^*(s') + s'$. which means that $\lambda^*(s) + s \leq \lambda^*(s') + s'$.

If $\lambda^*(s) + s - s' > \lambda_1(q)$. Let $\varepsilon > 0$ fixed such that $\lambda^*(s) + s - s' - \varepsilon > \lambda_1(q)$. Using the definition of $\lambda^*(s)$ there exist $\lambda > \lambda^*(s) - \varepsilon$ such that the problem (1) with $\gamma = \lambda + s$ and $\mu = \lambda$ possess a positive solution $u_\lambda \in C^1(\bar{\Omega})_+$. If we put $\mu = \lambda + s - s'$ then u_λ is a positive solution of (1) with $\gamma = \mu + s'$ and $\mu = \mu + s' - s$. Since $\mu + s' - s > \mu$ then u_λ is a positive super solution of (1) with $\gamma = \mu + s'$. As we have $\mu > \lambda^*(s) + s - s' - \varepsilon > \lambda_1(q)$ and ε is arbitrary then $\lambda^*(s') \geq \lambda^*(s) + s - s'$ then we obtain $\lambda^*(s) + s \leq \lambda^*(s') + s'$.

5- Since $\lambda^*(s)$ is bounded and non-increasing then for all $s \in \mathbb{R}$ we have

$$\lim_{s \rightarrow s'-0} \lambda^*(s) \geq \lambda^*(s') \geq \lim_{s \rightarrow s'+0} \lambda^*(s) \tag{10}$$

Also, $\lambda^*(s) + s$ is non-decreasing, then

$$\lim_{s \rightarrow s'-0} \lambda^*(s) + s \geq \lambda^*(s') + s' \leq \lim_{s \rightarrow s'+0} \lambda^*(s) + s$$

Then

$$\lim_{s \rightarrow s'-0} \lambda^*(s) \leq \lambda^*(s') \leq \lim_{s \rightarrow s'+0} \lambda^*(s) \tag{11}$$

Combining relation (10) and (11) we obtain that λ^* is continuous. ■

In (γ, μ) plane, we define de curve C as follows

$$C := \{(\lambda^*(s) + s, \lambda^*(s)), s \in \mathbb{R}\}$$

By the proposition 1, we can conclude that the curve C is locally finite, continuous and $C \subset [\lambda_1(p), +\infty) \times [\lambda_1(q), +\infty)$.

Now we will study the existence of positive solutions for the problem (1) in the region where (γ, μ) belong to $[\lambda_1(p), +\infty) \times [\lambda_1(q), +\infty)$. So in the case where $\lambda_1(p)$ and $\lambda_1(q)$ have the same eigenspace we have the following theorem.

Theorem 3 : Suppose that (4) is violated, then (1) has no solution expect for $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$.

Proof of Theorem 3 : Suppose that (4) is not satisfy and that $\gamma > \lambda_1(p)$ and $\mu > \lambda_1(q)$.

Assume that the problem (1) has a positive solution.

Using both relation (8) and (9) we have

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx + \beta_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma \\ & + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \left(\frac{\psi_p^p}{u^{p-1}} \right) dx + \beta_2 \int_{\partial\Omega} |u|^{q-2} u \left(\frac{\psi_p^p}{u^{p-1}} \right) d\sigma \\ & = \gamma \int_{\Omega} \psi_p^p dx + \mu \int_{\Omega} \psi_p^p u^{q-p} dx \leq \lambda_1(p) \int_{\Omega} \psi_p^p dx + \lambda_1(q) \int_{\Omega} \psi_p^p u^{q-p} dx. \end{aligned}$$

It's imply that $\mu = \lambda_1(q)$ and $\gamma = \lambda_1(p)$ which is a contradiction.

The proposition 1 ensure the existence of the positive solution of problem (1) for $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$. ■

In the case where $\lambda_1(p)$ and $\lambda_1(q)$ have a different eigenspaces, we have the following theorem.

Theorem 4 : Suppose that (4) is verified and that $\lambda > \lambda_1(p)$ is fixed. If we have $\lambda_1(q) < \mu < \lambda^*(\gamma - \mu)$ then (1) possesses at least one positive solution.

If $\mu > \lambda^*(\gamma - \mu)$, then (1) has not a positive solutions.

Proof of theorem 4 : Let's consider that $\mu > \lambda^*(\gamma - \mu)$ and with the definition of $\lambda^*(\gamma - \mu)$, (1) has no positives solution.

Presume that $\lambda_1(q) < \mu < \lambda^*(s)$ with $s = (\gamma - \mu)$ adding to that the definition of $\lambda^*(s)$, there exist $\lambda \in (\mu, \lambda^*(s))$ such that (1) with $\gamma = \lambda + s$ and $\mu = \lambda$ has a positive solution $w \in \text{int } C^1(\bar{\Omega})_+$ which conclude the result. ■

3. Conclusion

The most important statement in this work is that the existence of positive solution of (p, q) -Laplacian under the Robin boundary condition is depending

of parameters γ, μ and the first eigenvalue $\lambda_1(p)$ respectively $\lambda_1(q)$. For the existences results we have:

1. In the case where $\gamma > \lambda_1(p)$ and $0 < \mu < \lambda_1(q)$ or $0 < \gamma < \lambda_1(p)$ and $\mu > \lambda_1(q)$. The existence of positive solutions is established using the lemma 1.
2. For $\gamma > \lambda_1(p)$ and $\mu \leq 0$. Col theorem ensures the existence of positive solutions.
3. If we have $\gamma \leq 0$ and $\mu > \lambda_1(q)$ then the existence of positive solutions is given by the minimization theorem.
4. If (4) is verified and $\lambda_1(q) < \mu < \lambda^*(\gamma - \mu)$ then (1) possesses at least one positive solution.

Concerning the non existences results we have:

1. In the case where $\gamma \neq \lambda_1(p)$ and $\mu \neq \lambda_1(q)$ there is no non trivial solution.
2. If (4) is not verified then there are no positive solutions except for $\gamma = \lambda_1(p)$ and $\mu = \lambda_1(q)$.
3. If $\mu > \lambda^*(\gamma - \mu)$ and (4) is satisfied then we have no positive solution.

Conflict of interest

All data generated or analysed during this study are included in this published.

We state that there is no conflict nor competing interests.

Financing

The study was performed without financial support.

Data availability

Manuscript has no associated data.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

References

- [1] Wilhelmsson, H. Phys Rev.A 36, 965–966. (1987) . Explosive instabilities of reaction-diffusion equations,
- [2] Aris Pitman, R. Research Notes in Mathematics, Vol. 24, , London-San Francisco-Melbourne, 191 pp., (1978). Mathematical modelling techniques.
- [3] Cherfils, L. and Il'yasov, Y. Comm. Pure Appl. Anal. 4, 9–22 (2005), Anal. 154, 297–324 (2000). On the stationary solutions of generalized reaction diffusion equations with $\mathcal{L}_{p,q}$ -Laplacian,
- [4] Fife, P.C. Lect Notes in Biomathematics. 28, Springer-Verlag, Berlin-

Heidelberg-New York. IV, 185 S., 23 Abb., DM 21.50. (1979)

Mathematical aspects of reacting and diffusing systems.

[5] Benci, V., D'Avenia P., Fortunato, D. and Pisani, L. Arch. Rational Mech. Anal 154 297-324 (2000) . Solitons in several space dimensions Derrick's Problem and infinitely many solutions

[6] Benci, V., Fortunato, D. and Pisani, L. Rev. Math. Phys. 10,315–344 (1998). Soliton like solutions of a Lorentz invariantequation in dimension 3,

[7] Benci, V., Micheletti, A.M. and Visetti, D. Topol. Methods Nonlinear Anal. 17, 191–211 (2001) . An eigenvalue problem for a quasilinear elliptic field equation on \mathbb{R}^n

[8] Derrick, G.H. J. Math. Phys. 5, 1252–1254 (1964) ., Comments on nonlinear wave equations as models for elementary particles,

[9] Motreanu, D. and Tanaka, M. Minimax Theory and its Applications 01 001-020 (2016). On positive solution for (p,q) -Laplace equation with indefinite weight,

[10] Kajikiya, R., Tanaka, M. and Tanaka, S., Electronic Journal of Differential Equations, Vol 2017, No. 107, pp. 1-37. (2017). Bifurcation of positive solutions for the one dimensional (p, q) -Laplace equation

[11] Merzagui., (2020). Méthode de sous et sur solutions pour la résolution des problèmes elliptiques,

[12] Pucci, P., Serrin, J., Springer, (2007), vol 73. The maximum principle,

[13] Bobkov, V. and Tanaka, M. arXiv:2004.02928v2 [math.AP] (2021).

Generalized Picone inequalities and their application to (p,q) -Laplace equation.