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Approximation of Fixed Points for Generalized Nonexpansive Maps and Application

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Abstract

In this article, we demonstrate strong convergence results of a map on a closed, bounded and convex subset of a CAT(0) space, satisfying the (RCSC)-condition using AP iterative scheme. By using the AP iterative scheme, we can find the solution of differential and integral equations. The outcome we acquire improves and broadens numerous recent findings in the literature. Finally, a numerical example is also provided of a mapping which pleases the (RCSC)-condition but fails to meet condition (C).

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1 Introduction

The initials of the term CAT personalizes Cartan, Alexandrov and Toponogov, who has made essential contributions to the understanding of metric spaces having non positive curvature. “Let

(E, d) be a metric space and $a, b \in E$ with $d(a, b) = l$. A geodesic path from a to b is a isometry $c : [0, l] \to E$ such that $c(0) = a$ and $c(l) = b$. The image of a geodesic path is called a geodesic segment. A metric space E is a (uniquely) geodesic space, if every two points of E are joined by only one geodesic segment. A geodesic triangle $\Delta(a_1, a_2, a_3)$ in a geodesic space E consists of three points a_1, a_2, a_3 of E and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(a_1, a_2, a_3)$ is the triangle $\overline{\Delta}(a_1, a_2, a_3) := \Delta(\overline{a_1}, \overline{a_2}, \overline{a_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(a_i, a_j) = d_{\mathbb{R}^2}(\overline{a_i}, \overline{a_j})$, $\forall$ i, j = 1,2,3. A geodesic space E is a CAT(0) space, if for each geodesic triangle $\Delta(a_1, a_2, a_3)$ in E and its comparison triangle $\overline{\Delta} := \Delta(\overline{a_1}, \overline{a_2}, \overline{a_3})$ in $\mathbb{R}^2$, the CAT(0) inequality $d(a, b) = d_{\mathbb{R}^2}(\overline{a}, \overline{b})$ is satisfied for all $a, b \in \Delta$ and $\overline{a}, \overline{b} \in \overline{\Delta}$." [1, 2, 6] provide a detailed discussion of these spaces and their importance in various branches of mathematics are given .

Kirk [9, 10], a renowned mathematician developed a more general result to investigate the invariant point outcomes in CAT(0) space. He demonstrated, among other things, that every nonexpansive map described on a closed bounded convex subset of a complete CAT(0) space has an invariant point. Since that time, the invariant point results of various maps and iterative methods in a CAT(0) space has advanced quickly, and numerous papers have been published ([4, 5, 8, 12, 13]). The iterative scheme theory is very rich in one or more mappings studied intensively.It is the main tool for analyzing the various mappings.

Consider $K(\neq \phi)$ is a subset of CAT(0) space E. A self map S on K is a nonexpansive map whenever $d(Sa, Sb) \le d(a, b)$ for all $a, b \in K$. It is quasi-nonexpansive map if $F(S) \neq \phi$ and $d(Sa, p) \le d(a, p)$ $\forall$ $a \in K$ and $p \in F(S)$, where $F(S)$ is the set of invariant points of S. Many mathematicians have recently considered a number of extensions and generalizations of nonexpansive maps. Suzuki[14] invented the idea of generalized nonexpansive maps and demonstrated a few existence and convergence axioms for these maps. A self map S on K pleases condition(C) if $\frac{1}{2}d(a, Sa) \le d(a, b)$ implies $d(Sa, Sb) \le d(a, b)$ $\forall$ $a, b \in K$.

Karapınar [7] proposed a new modification of maps pleasing condition(C) to maps satisfying (RCSC)-condition in 2013.

A map $S : K \to K$ is said to meet Reich-Chatterjea-Suzuki-(C) condition ((RCSC)-condition) if $\frac{1}{2}d(a, Sa) \le d(a, b)$ implies $d(Sa, Sb) \le \frac{1}{3}(d(a, b) + d(Sa, b) + d(a, Sb))$ for all $a, b \in K$.

Banach is acclaimed with putting the concepts of fixed point results into an abstract structure that is applicable to a wide range of applications by introducing Picard iteration in 1922.

Many physical problems of engineering and applied sciences are mostly constructed in the form of fixed point equations. We can only approximate the solution which becomes very relevant and this necessitated various iterative schemes. Recently, Lamba and Panwar [11] established approximation outcomes using a new iteration process, we will call it AP iterative method in the framework of CAT(0) space, with the claim that it is even faster than many iterative methods like M, K and Thakur new iteration process, as follows

$$a_0 \in T$$
$$c_n = S((1-\tilde{\beta}_n)a_n \oplus \tilde{\beta}_n S a_n)$$
$$b_n = S((1-\alpha_n)Sa_n \oplus \alpha_n Sc_n)$$
$$a_{n+1} = Sb_n. \tag{1.1}$$

Empowered by the preceding works, we demonstrate a few strong convergence results of generalized nonexpansive maps in CAT(0) space in this article. Our findings generalize the outcomes of several others in the literature.

2 Preliminaries

To keep things simple, we review a few definitions and exceptions.

Lemma 2.1 *[3] "Consider E is a CAT(0) space. For $a, b, c \in E$ and $h \in [0,1]$ we have* $d((1-h)a \bigoplus hb, c) \leq (1-h)d(a,c) + hd(b,c)$."

Lemma 2.2 *[3] "For $a, b, c \in E$ and $h \in [0,1]$. Then* $d((1-h)a \bigoplus hb, c)^2 \leq (1-h)d(a,c)^2 + hd(b,c)^2 - h(1-h)d(a,b)^2$."

Lemma 2.3 *[15] "Suppose $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative numbers pleasing the inequality*

$a_{n+1} \leq (1+b_n)a_n + c_n$ *for all* $n \geq 1$.

If $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists."*

Proposition 2.4 *Presume $K (\neq \phi)$ is a subset of a CAT(0) space E and S be self map on K pleasing (RCSC)-condition. S is a quasi-nonexpansive map if S has a invariant point.*

Proof. Presume S has a invariant point, i.e., $c \in F(S)$. Simply, $0 = \frac{1}{2}d(c, Sc) \leq d(c, b)$ implying

$$d(Sc, Sb) \leq \frac{1}{3}[d(c, b) + d(Sc, b) + d(c, Sb)]$$
$$\leq \frac{1}{3}[2d(c, b) + d(Sc, Sb)]$$

Thus, $d(c, Sb) = d(Sc, Sb) \leq d(c, b)$ which completes the proof. □

Lemma 2.5 *Presume $K(\neq \phi)$ is subset of a CAT(0) space E and S be a self map on K pleasing (RCSC)-condition. Then, F(S) is closed.*

Proof. Consider $\{c_n\}$ is a sequence in F(S) and converges to a point $c \in K$.

It is obvious that $\frac{1}{2}d(c_n, Sc_n) = 0 = d(c_n, c)$ for $n \in N$.

Thus, we acquire
$$\lim_{n\to\infty} d(c_n, Sc) = \lim_{n\to\infty} d(Sc_n, Sc)$$
$$\leq \lim_{n\to\infty} \frac{1}{3}[d(c_n, c) + d(Sc_n, c) + d(c_n, Sc)]$$
$$= \lim_{n\to\infty} \frac{1}{3}[2d(c_n, c) + d(c_n, Sc)]$$
$$\Rightarrow \limsup_{n\to\infty} d(c_n, Sc) \leq \limsup_{n\to\infty} d(c_n, c) = 0,$$

then, $\{c_n\}$ converges to Sc. By uniqueness of limit, Sc = c. Hence, $c \in F(S)$, i.e., F(S) is closed. □

Lemma 2.6 *Consider $K(\neq \phi)$ is a subset of a CAT(0) space E and S is self map on K pleasing (RCSC)-condition. Then, for all $a, b \in K$:*

1. $d(Sa, S^2a) \leq d(a, Sa)$.
2. *Either* $\frac{1}{2}d(a, Sa) \leq d(a, b)$ *or* $\frac{1}{2}d(Sa, S^2a) \leq d(Sa, b)$.
3. *Either* $d(Sa, Sb) \leq \alpha d(Sa, b) + \alpha d(a, Sb) + (1 - 2\alpha)d(a, b)$
   or $d(S^2a, Sb) \leq \alpha d(Sa, Sb) + \alpha d(S^2a, b) + (1 - 2\alpha)d(Sa, b)$.

Proof. The first assertion is a result of the (RCSC)-condition. Evidently, we get $\frac{1}{2}d(a, Sa) \leq d(a, Sa)$ which yields

$$d(Sa, S^2a) \leq \frac{1}{3}[d(a, Sa) + d(Sa, Sa) + d(S^2a, a)]$$
$$= \frac{1}{3}[d(a, Sa) + d(S^2a, a)]$$
$$\leq \frac{1}{3}[d(a, Sa) + d(Sa, a) + d(S^2a, Sa)]$$
$$\leq \frac{1}{3}[2d(a, Sa) + d(S^2a, Sa)]$$

that implies (1).

It is obvious that (3) is a result of (2). To demonstrate (2), presume the opposite, i.e., $\frac{1}{2}d(a, Sa) > d(a, b)$ and $\frac{1}{2}d(Sa, S^2a) > d(Sa, b)$ holds $\forall\ a, b \in K$.

Using triangle inequality and (1), we acquire

$$\begin{aligned} d(a, Sa) &\leq d(a, b) + d(b, Sa) \\ &< \frac{1}{2}d(a, Sa) + \frac{1}{2}d(Sa, S^2a) \\ &\leq \frac{1}{2}d(a, Sa) + \frac{1}{2}d(Sa, Sa) = d(a, Sa) \end{aligned}$$

which is contradiction. As a result, we get (2). □

3 Main results

Lemma 3.1 *Presume S is a map on a nonempty, bounded closed convex subset of a complete CAT(0) space such that $F(S) \neq \phi$. Assume that S satisfies (RCSC)-condition. Let $\{a_n\}$ be defined by AP iteration process. Then $\lim_{n\to\infty} d(a_n, \tilde{p})$ exists $\forall\ \tilde{p} \in F(S)$.*

Proof. Using Proposition 2.4, we acquire

$$\begin{aligned} d(c_n, \tilde{p}) &= d\left(S((1-\tilde{\beta}_n)a_n \oplus \tilde{\beta}_n Sa_n), \tilde{p}\right) \\ &\leq d\left((1-\tilde{\beta}_n)a_n \oplus \tilde{\beta}_n Sa_n, \tilde{p}\right) \\ &\leq (1-\tilde{\beta}_n)d(a_n, \tilde{p}) + \tilde{\beta}_n d(Sa_n, \tilde{p}) \\ &\leq (1-\tilde{\beta}_n)d(a_n, \tilde{p}) + \tilde{\beta}_n d(a_n, \tilde{p}) \\ &= d(a_n, \tilde{p}) \end{aligned}$$

for all $n \in N$. Also, we have

$$d(b_n, \tilde{p}) = d(S((1-\tilde{\alpha}_n)Sa_n \oplus \tilde{\alpha}_n Sc_n), \tilde{p})$$
$$\leq d((1-\tilde{\alpha}_n)Sa_n \oplus \tilde{\alpha}_n Sc_n, \tilde{p})$$
$$\leq (1-\tilde{\alpha}_n)d(Sa_n, \tilde{p}) + \tilde{\alpha}_n d(Sc_n, \tilde{p})$$
$$\leq (1-\tilde{\alpha}_n)d(a_n, \tilde{p}) + \tilde{\alpha}_n d(a_n, \tilde{p})$$
$$\leq (1-\tilde{\alpha}_n)d(a_n, \tilde{p}) + \tilde{\alpha}_n d(a_n, \tilde{p})$$
$$= d(a_n, \tilde{p})$$

for all $n \in N$. Likewise, we acquire

$$d(a_{n+1}, \tilde{p}) = d(Sb_n, \tilde{p})$$
$$\leq d(b_n, \tilde{p})$$
$$\leq d(a_n, \tilde{p})$$

consequently, we have $d(a_{n+1}, \tilde{p}) \leq d(a_n, \tilde{p})$ $\forall\ n \geq 1$. As a result, $\lim_{n\to\infty} d(a_n, \tilde{p})$ exists. So, $\{a_n\}$ is bounded. □

Theorem 3.2 *Let E, S and* $\{a_n\}$ *be the same as in Lemma 3.1 such that* $F(S) \neq \phi$. *Presume* $K (\neq \phi)$ *is a compact convex subset of E. If* $\lim_{n\to\infty} d(Sa_n, a_n) = 0$, *then* $\{a_n\}$ *converges to a invariant point of S if and only if* $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$.

Proof. If the sequence $\{a_n\}$ converges to a point $\tilde{p} \in F(S)$, then

$\lim_{n\to\infty} d(a_n, \tilde{p}) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d(a_n, F(S)) = 0$

and therefore, $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$. Assume for the converse part that $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$.

By Lemma 3.1, we get

$d(a_{n+1}, \tilde{p}) \leq d(a_n, \tilde{p})$ for any $\tilde{p} \in F(S)$

Therefore, we acquire $d(a_{n+1}, F(S)) \leq d(a_n, F(S))$. Thus, we achieve $\lim_{n\to\infty} d(a_n, F(S))$ exists. Because $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$ therefore $\lim_{n\to\infty} d(a_n, F(S)) = 0$.

Next, we demonstrate $\{a_n\}$ is a Cauchy sequence in K. Consider $\epsilon > 0$ is arbitrarily chosen.

As $\lim_{n\to\infty} d(a_n, F(S)) = 0$, there is n_0 such as $\forall\ n \geq n_0$, we have

$d(a_n, F(S)) < \frac{\epsilon}{4}$.

More specifically, $\inf\{d(a_{n_0}, \tilde{p}) : \tilde{p} \in F(S)\} < \frac{\epsilon}{4}$,

so there must be a $\tilde{p} \in F(S)$ such as

$d(a_{n_0}, \tilde{p}) < \frac{\epsilon}{2}$.

Thus, for $m, n \geq n_0$, we attain

$d(a_{n+m}, a_n) \leq d(a_{n+m}, \tilde{p}) + d(a_n, \tilde{p})$

$\leq 2d(a_{n_0}, \tilde{p}) < 2.\frac{\epsilon}{2} = \epsilon$ which shows that $\{a_n\}$ is a Cauchy sequence . Since K is a closed subset of a complete metric space and therefore $\{a_n\}$ must converge to some a in K. As $\lim_{n\to\infty} d(a_n, F(S)) = 0$ which gives $d(a, F(S)) = 0$. As F(S) is closed, we must have $a \in F(S)$ □

Example 3.3. Describe a map $S : [0, 1] \to [0, 1]$ as

$$Sa = \begin{cases} 0, & a \in [0, \frac{2}{3}), \\ \frac{a}{2}, & a \in [\frac{2}{3}, 1]. \end{cases}$$

Set $a = \frac{5}{6}$, $b = 0.423$, we see that

$\frac{1}{2}d(a, Sa) = |Sa - Sb| = \frac{1}{2}|\frac{5}{6} - \frac{5}{12}| = \frac{5}{24} < |\frac{5}{6} - 0.423| = 0.410 = d(a, b)$.

and

$$d(Sa, Sb) = |\frac{5}{12}| = 0.416 > d(a, b),$$

i.e.,

$$\frac{1}{2}d(a, Sa) \leq d(a, b) \not\Rightarrow d(Sa, Sb) \leq d(a, b),$$

hence, S fails to meet condition (C).

Assume the following scenarios to ensure that S meets condition (RCSC):

Case I: Let $a \in [0, \frac{2}{3})$, then

$d(Sa, Sb) = 0 \leq \frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb)]$.

Case II: Let $a, b \in [\frac{2}{3}, 1]$, then

$$d(Sa, Sb) = |\frac{a}{2} - \frac{b}{2}|.$$

Since

$$d(a, b) = |a - b| > |\frac{a}{2} - \frac{b}{2}| = d(Sa, Sb),$$

$$d(Sa, b) = |\frac{a}{2} - b| > |\frac{a}{2} - \frac{b}{2}| = d(Sa, Sb)$$

and

$$d(a, Sb) = |a - \frac{b}{2}| > |\frac{a}{2} - \frac{b}{2}| = d(Sa, Sb),$$

which implies that

$d(Sa, Sb) \leq \frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb)]$

for all $a, b \in [\frac{2}{3}, 1]$

Case III: Let $a \in [0, \frac{2}{3})$ and $b \in [\frac{2}{3}, 1]$ or $b \in [0, \frac{2}{3})$ and $a \in [\frac{2}{3}, 1]$ Then

$$d(Sa, Sb) = \frac{b}{2}.$$

Also,

$$\frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb) = \frac{1}{3} b - a + b + |a - \frac{b}{2}| .$$

Next, we have two subcases which are as follows

Case III(A): $a \geq \frac{b}{2}$, then $d(a, b) = a - \frac{b}{2}$, so we have

$$\frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb) = \frac{b}{2} = d(Sa, Sb).$$

Case III(B): $a < \frac{b}{2}$, then $d(a, \frac{b}{2}) = \frac{b}{2} - a$,

$$\frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb) = \frac{1}{3} \frac{5b}{2} - b = \frac{b}{2} = d(Sa, Sb).$$

Hence $d(Sa, Sb) \leq \frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb)]$ for all $a \in [0, \frac{2}{3})$ and $b \in [\frac{2}{3}, 1]$.

Case IV: Also, for $b \in [0, \frac{2}{3})$ and $a \in [\frac{2}{3}, 1]$. we can easily derive by swapping the roles of a and b in Case III

$d(Sa, Sb) \leq \frac{1}{3}[d(a, b) + d(Sa, b) + d(a, Sb)]$ for all $b \in [0, \frac{2}{3})$ and $a \in [\frac{2}{3}, 1]$.

Hence, we can admit that S meets condition (RCSC) for all $a, b \in K$.

4 Conclusion

The extension of the linear version of convergence outcomess to nonlinear spaces has its own importance. We impose a linear version of convergence outcomes to the maps which satisfies condition (RCSC) for AP iterative method in the setting of Banach space to nonlinear CAT(0) spaces. Also, we gave an example to illustrate the facts.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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