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A Covering Property via Generalized Fuzzy Preopen Sets

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Abstract

This study presents and analyzes the idea of fuzzy μ-precompact spaces. The investigation is based on the result that, in generalized fuzzy topological spaces, every fuzzy μ- preopen set contains a fuzzy μ-open subset. It is established that fuzzy μ-precompactness is a weaker notion than fuzzy μ-compactness but stronger than weakly fuzzy μ-paracompactness in the framework of generalized fuzzy topological spaces.

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1) INTRODUCTION

In 1965, Zedeh [4] established the foundation of fuzzy set theory. Shortly thereafter, in 1968, Chang[1] introduced the concept of fuzzy topology. Within the realm of classical Topology, Csa'sza'r[3] introduced the idea of generalized topology. Let I^X denote the collection of all subsets of a nonempty set X. A fuzzy subcollection μ of I^X is referred to as a generalized fuzzy topology on X [2] if it satisfies two conditions: first, $0_X \in \mu$, and second, for any collection $\{\lambda_\alpha : \alpha \in \Delta\}$ of elements in μ the supremum $\vee_{\alpha \epsilon \Delta} \lambda_\alpha$ must also being to μ. A generalized fuzzy topological space is defined as a nonempty set X equipped with a generalized fuzzy topology μ. And is denoted by (X, μ). A generalized fuzzy topological space is abbreviated as GFTS. A fuzzy set $\lambda \in \mu$ is called a fuzzy μ- open set in (X, μ) [2]. The complement of a fuzzy μ- open set is termed a fuzzy μ- closed set in (X, μ).

For fuzzy subset λ of X, the generalized fuzzy closure, denoted by $c_\mu(\lambda)$, is defined as the intersection of all fuzzy μ- Closed sets that contain λ. In a similar manner, the generalized fuzzy interior of a fuzzy subset λ, denoted by $i_\mu(\lambda)$, is the union of all fuzzy μ- open sets that are contained within λ. Furthermore, a fuzzy subset λ of X is said to be μ- open (or fuzzy μ- closed) if and only if λ coincides with $i_\mu(\lambda)$, (or $\lambda = c_\mu\, i_\mu(\lambda)$). Additionally, for any fuzzy subset λ of X, the relationship $c_\mu(\lambda) = 1_X - i_\mu(\lambda)$ holds true.

This paper explores additional properties of fuzzy μ- preopen sets in generalized fuzzy topological spaces (GFTS) or fuzzy μ- spaces.

Throughout the paper X stands for generalized fuzzy topological space (X, μ) or GFTS.

2) A Covering Property with Generalized Fuzzy Preopen Sets.

In order to present the main findings, we revisit fundamental definitions and results.

Definition 2.1. [2] A fuzzy subset ζ of X is called

(i) fuzzy μ-preopen if $\zeta \subset i_\mu\big(c_\mu(\zeta)\big)$.

(ii) fuzzy μ-semiopen if $\zeta \subset c_\mu\big(i_\mu(\zeta)\big)$.

(iii) fuzzy μ- regular open if $\zeta \subset i_\mu\big(c_\mu(\zeta)\big)$.

We witness that fuzzy set ζ of X is fuzzy μ-preopen (resp. fuzzy μ-semiopen) iff there exists a fuzzy μ-open set β such that $\zeta \subset \beta \subset c_\mu(\zeta)$ (resp. $\beta \subset \zeta \subset c_\mu(\beta)$)

Definition 2.2. A fuzzy μ- cover of X is a collection ζ of fuzzy μ- subsets of X such that $\bigcup_{\zeta\in\zeta} \zeta = X$. $\mathfrak{A}$ is called a fuzzy μ-open cover (resp., fuzzy μ-preopen cover) of X if $\mathfrak{A}$ is a fuzzy μ-open collection (resp. fuzzy μ-preopen collection) of X and fuzzy μ- covers of X.

Definition 2.3. A fuzzy μ- space is called fuzzy μ- compact if each fuzzy μ-open cover of X has a finite fuzzy μ- subcover.

Definition 2.4. A fuzzy μ- space is said to be weakly fuzzy μ- compact (also referred to as fuzzy ω_μ- compact) if, for every fuzzy μ- open cover $\mathcal{M}$ of X, there exists a finite subcollection, $\mathbb{Z}_n \subseteq \mathcal{M}$ such that $\bigcup_{G\in\mathbb{Z}_n} c_\mu(G) = X$.

Definition 2.5. A fuzzy μ- space is called fuzzy μ- S- closed if for every fuzzy μ-semiopen cover $\mathfrak{D}$ of X, there exists a finite subfamily $\mathfrak{D}_n \subseteq \mathfrak{D}$ such that $\bigcup_{\beta\in\mathfrak{D}_n} c_\mu(\beta) = X$

We are about to introduce the following:

Definitiona 2.6. Let $\mathfrak{S}$ be a fuzzy μ- preopen collection in a fuzzy μ - space X. For each $\zeta \in \mathfrak{S}$, suppose there exists a fuzzy μ -open set ξ such that $\zeta \subset \xi \subset c_\mu(\zeta)$. Define the collection $\mathcal{G} = \{\xi \mid \zeta \in \mathfrak{S}, \zeta \subset \xi \subset c_\mu(\zeta)\}$. Then, $\mathcal{G}$ is referred to as a fuzzy μ - open super collection of $\mathfrak{S}$.

Hence, for any fuzzy μ - preopen collection $\mathfrak{S}$ in a fuzzy μ -space X, there exists a fuzzy μ - open collection $\mathcal{G}$ that serves as a super collection of $\mathfrak{S}$. Additionally, if $\mathfrak{S}$ constitutes a fuzzy μ - cover of X, then the corresponding collection $\mathcal{G}$ also acts as a fuzzy μ - cover of X. In this scenario, $\mathcal{G}$ is known as a fuzzy μ - open super cover of the fuzzy μ - preopen cover $\mathfrak{S}$.

Definition 2.7. A fuzzy μ - space X is said to be fuzzy μ - precompact if every fuzzy μ - preopen cover of X possesses a finite fuzzy μ – open super cover.

Suppose $\mathcal{G}$ is a finite fuzzy μ-open super cover corresponding to an fuzzy μ-preopen cover $\mathfrak{S}$ of a fuzzy μ- precompact space X. Then, for each $\xi \in \mathfrak{S}$ such that $\zeta \subset \xi \subset c_\mu(\zeta)$. Consequently, one can derive a finite fuzzy μ-subcollection of $\mathfrak{S}$ given by $\{\zeta \mid \xi \in \mathcal{G}, \zeta \subset \xi \subset Cl(\zeta)\}$.

It is important to observe that every fuzzy μ- compact space is also fuzzy μ-precompact, and every fuzzy μ- precompact space is weakly fuzzy μ - compact. However, the converses of these relationships do not necessarily hold.

Example 2.1. Let $\mathbb{U}$ = [0, +∞) and for each $x \in U$, let

Let $\Omega_x(y) = \begin{cases} e^{-y} & if\ 0 < y < x \\ 0\ if\ y \leq x \end{cases}$

$$\Omega_\infty(y) = e^{-y}, \forall \in U$$

$$\mu = \{\Omega_\infty\} \cup \{\Omega_x(w), \forall \in U\}$$

Let (X, μ) be a GFTS. Clearly the space is fuzzy μ-precompact but not fuzzy μ-compact space.

Lemma 2.2. If λ is fuzzy μ-preopen in X, then $i_\mu(c_\mu(\lambda))$ is μ- fuzzy regularly μ- open in X.

Proof : Since λ is a fuzzy μ-preopen set in X, there exists a fuzzy μ-open set β such that

$$\lambda \subset \beta \subset c_\mu(\lambda) \text{ and } c_\mu(\lambda) = c_\mu(\beta)$$

Applying the interior operator i_μ to both sides of the closure equality gives

$$i_\mu(c_\mu(\lambda)) = i_\mu(c_\mu(\beta)).$$

Since $i_\mu(c_\mu(\beta))$ is fuzzy regularly μ - open, it follows that $i_\mu(c_\mu(\lambda))$ is also fuzzy regularly μ- open in X.

Example 2.3. Let $\mathbb{U}$ = [0, +∞) and for each $x \in U$, let

Let $\Omega_x(y) = \begin{cases} e^{-y} & if\ 0 < y < x \\ 0\ if\ y \leq x \end{cases}$

$$\Omega_x(z) = \begin{cases} e^{-z} & if\ \ 0 < z < y < x \\ & 0\ if\ y \leq x \end{cases}$$

$$\Omega_x(w) = \begin{cases} e^{-w}\ if\ 0 < y < w < x \\ 0\ if\ y \leq x \end{cases}$$

$$\Omega_\infty(y) = e^{-y}, \forall \in U$$

$$\mu = \{\Omega_\infty\} \cup \{\Omega_x(w), \forall \in U\}$$

Let (X, μ) be a GFTS. It is observed that for fuzzy open set $\Omega_x(z)$, $i_\mu(c_\mu(\Omega_x(z))) = \Omega_x(w)$ that is $\Omega_x(z)$ is fuzzy μ - regularly in (X, μ). Since $\Omega_x(z) \not\subset i_\mu(c_\mu(\Omega_x(z)))$. Then $\Omega_x(z)$, is not fuzzy μ - preopen set in (X, μ).

Theorem 2.4. A fuzzy μ- space X is fuzzy μ - precompact iff for every fuzzy μ-preopen cover $\mathfrak{S}$ of X, there exists a finite fuzzy μ - subcollection $\mathcal{D} \sqsubseteq \mathfrak{S}$ such that the family $\{i_\mu(c_\mu(\lambda)) | \lambda \in \mathcal{D}\}$ forms a fuzzy regularly μ - open supercover of X.

Proof: By the fuzzy μ - precompactness of X, there exists a finite fuzzy μ-open super cover, $\mathcal{M}$ of the fuzzy μ-open cover $\mathfrak{S}$. For every fuzzy set $\beta \in \mathcal{M}$, one can find a set $\lambda \in \mathfrak{S}$ such that

$$\lambda \subset \beta \subset c_{\mu}(\lambda)$$

which further implies

$$\lambda \subset \beta \subset i_{\mu}(c_{\mu}(\lambda)) \subset c_{\mu}(\lambda)$$

Define

$\mathcal{D} = \{\lambda \in \mathfrak{S} \,|\, for\ some\ \beta \in \boldsymbol{\mathcal{M}},\ \lambda \subset \beta \subset c_{\mu}(\lambda)\}$.

This shows that $\mathcal{D}$ is a finite fuzzy μ- subcollection of $\mathfrak{S}$. Since $\boldsymbol{\mathcal{M}}$ is a fuzzy μ - cover of X, the collection $\{i_{\mu}(c_{\mu}(\lambda)) \,|\, \lambda \in \mathcal{D}\}$ also forms a fuzzy μ - cover X. By Lemma 2.1. each set $i_{\mu}(c_{\mu}(\gamma))$ with $\gamma \in \mathcal{D}$ is fuzzy regular μ - open. Therefore, $\mathcal{D}$ is a finite fuzzy μ - subcollection of $\mathfrak{S}$ such that

$$\{i_{\mu}(c_{\mu}(\gamma)) \,|\, \gamma \in \mathcal{D}\}$$

Is a fuzzy regularly μ - open super cover of the fuzzy μ - preopen cover $\mathfrak{S}$ of X.

Conversely, suppose $\{i_{\mu}(c_{\mu}(\lambda)) \,|\, \lambda \in \mathcal{D}\}$ is such a cover. Since $i_{\mu}(c_{\mu}(\lambda))$ is fuzzy μ - open and satisfies

$\lambda \subset i_{\mu}(c_{\mu}(\lambda)) \subset c_{\mu}(\lambda)$,

For every $\lambda \in \mathcal{D}$ it follows that this collection forms a finite fuzzy μ - open super cover of $\mathfrak{S}$. Hence, X is fuzzy μ - precompact.

Theorem 2.5. The following statements are equivalent for a fuzzy μ- space X:

(a) The space X is fuzzy μ - precompact.

(b) For any X_{μ} - PO cover $\mathfrak{A}$ of X, there exists a finite fuzzy μ- subcollection $\vartheta \subseteq \mathfrak{A}$ such that the collection $\{i_{\mu}(c_{\mu}(\xi)) \,|\, \xi \in \vartheta\}$ forms a cover of X.

(c) If $\mathfrak{L}$ is a family of fuzzy μ - preclosed subsets of X satisfying $\bigcap_{E \in \mathfrak{L}} \Omega = 0_X$, then there exists a finite fuzzy μ - subfamily $\mathcal{F} \subseteq \mathfrak{L}$ such that $\bigcap_{X \in \mathcal{F}} i_{\mu}(c_{\mu}(X)) = 0_X$.

Proof: (a) ⇒ (b) : Proceed by Theorem 2.3.

(b) ⇒ (c) : Let $\mathfrak{L} = \{\Omega_{\xi} \,|\, \xi \in \Delta\}$ be a collection of fuzzy μ - preclosed sets such that $\bigcap_{\xi \in \Delta} \Omega_{\xi} = 0_X$

. It means that $\{1_X - \Omega_{\xi} \,|\, \xi \in \Delta\}$ is a X_{μ} - PO cover of X. By (b) , we acquire a finite fuzzy μ- subcollection $\left\{1_X - \Omega_{\xi_j} \,\middle|\, \xi_j \in \Delta, j \in \{1, 2, \dots\dots\dots\dots, n\}\right\}$ of $\{X - \Omega_{\xi} \,|\, \xi \in \Delta\}$ such that $\left\{i_\mu\left(c_\mu\left(X-\Omega_{\xi_j}\right)\right)\middle| j\in\{1,2,\ldots\ldots\ldots\ldots,n\}\right\}$ covers X. It denotes that X - $\bigcup_{k=1}^{n} i_\mu\left(c_\mu\left(X-\Omega_{\xi_j}\right)\right)=0_X$ and hence $\bigcap_{k=1}^{n} c_\mu\left(i_\mu\left(X-\Omega_{\xi_j}\right)\right)=0_X$.

(c) ⇒ (a): *Let X accomplish condition* (c). Suppose $\mathcal{W}=\left\{w_\xi\middle|\xi\in\Delta\right\}$ is a X_μ - PO cover of X. Consequently, we determine that $\mathfrak{L}=\left\{X-w_\xi\middle|\xi\in\Delta\right\}$ is a collection of fuzzy μ - preclosed sets such that $\bigcap\left\{X-w_\xi\middle|\xi\in\Delta\right\}=0_X$. By (c) we acquire a finite fuzzy μ - subcollection $\left\{X-w_{\xi_j}\middle|\xi_j\in\Delta\right\}, j\in\{1,2,\ldots\ldots\ldots\ldots,n\}$ such that $\bigcap_{k=1}^{n} c_\mu\left(i_\mu\left(X-w_{\xi_j}\right)\right)=0_X$. gives that $\bigcup_{k=1}^{n} i_\mu\left(c_\mu\left(w_{\xi_j}\right)\right)=X$. So, $\left\{w_{\xi_j}\middle|\xi_j\in\Delta,\right\}$, $j\in\{1,2,\ldots\ldots\ldots\ldots,n\}$ is a finite fuzzy μ - subcollection $\mathcal{W}$ *such that* $\left\{\left\{i_\mu\left(c_\mu\left(w_{\xi_j}\right)\right)\middle|\xi_j\in\Delta,\right\}\quad j\in\{1,2,\ldots\ldots\ldots\ldots,n\}\right\}$ covers X. Now by Theorem 2.4. is fuzzy μ - precompact.

Definition 2.8. A collection $\mathfrak{S}$ of fuzzy μ - subsets of X is called a fuzzy proximate μ - cover of X if it satisfies the condition $c_\mu(\bigcup_{C\in\mathfrak{S}}\ \Phi)$ = X.

Theorem 2.6. Every fuzzy μ-preopen cover of a fuzzy μ - precompact space X admits a finite fuzzy promixate μ - preopen subcover.

Proof. Let $\mathfrak{B}$={$\Phi_a|\alpha\in\Delta$} be a X_μ-PO cover of the fully μ -precompact space X. Since X is fully μ-precompact, there exists a finite family $\{\Pi_1, \Pi_2,\ldots\ldots \Pi_n\}$ of fuzzy μ-open sets forming a supercover of $\mathfrak{B}$

For each k ∈{1, 2,………,n}, there exists some $\alpha_k\epsilon\Delta$ such that

$\Phi_{\alpha_k}\subseteq\Pi_k\subseteq c_\mu(\Phi_{\alpha_k})$

Since $\{\Pi_1, \Pi_2,\ldots\ldots \Pi_n\}$ covers X, it follow that

$X=\ \bigcup_{k=1}^{n} c_\mu(\Phi_{\alpha_k})$ =($c_\mu \bigcup_{k=1}^{n}\Phi_{\alpha_k}$)

Hence, the collection ({Φ_1, Φ_2,…… Φ_n}) forms a finite fuzzy proximate μ-preopen cover of X.

Definition 2.9. A fuzzy μ-space X is said to be fuzzy extremally μ-disconnected. If, for every fuzzy μ-open set β in X, the μ-closure $c_\mu(\beta)$ is also fuzzy μ-open.

Theorem 2.7. A fuzzy ω_μ-compact and fuzzy extremally μ-disconnected space is a fuzzy μ-precompact space.

Proof : Let $\mathbb{Z}$ = {Ω_ξ}|$\xi\epsilon\Delta$} be a X_μ -PO cover of a fuzzy extremally μ-disconnected space X that is ω_μ-compact. For each $\xi\epsilon\Delta$, there exists a fuzzy μ − open set Π_ξ such that

$\Omega_\xi\subset\Pi_\xi\subset c_\mu\ \Omega_\xi$ = c_μ (Π_ξ).

Clearly, the collecting $\mathcal{G}$ = {Π_ξ| $\xi\in\Delta$} be a X_μ-O cover of X. Since X is fuzzy ω_μ - compact, there exists a finite fuzzy μ -sub collection {Π_ξ, |$\xi_j\in\Delta$, j ∈ {1,2, … … . , n}}

{$c_\mu\left(\Pi_{\xi_j}\right)$ |$\xi_j\in\Delta$, j ∈ {1,2, … . . , n}}

Covers X. Due to the fuzzy external μ-disconnectedness of X, we observe that the collection.

{$c_\mu\left(\Pi_{\xi_j}\right)$ |$\xi_j\in\Delta$, j ∈ {1,2, … . . , n}} is a finite fuzzy μ-preopen super cover $\mathbb{Z}$.

Definition 2.10. A fully μ-semiopen set λ in X is said to be fuzzy μ- covered if $\beta \subset \lambda \subset C_{\mu}(\beta)$ for some fuzzy μ-open set β, then there exists a fuzzy μ-open set γ such that $\beta \subset \lambda \subset \gamma \subset c_{\mu}(\beta)$.

Lemma 2.8. A fuzzy covered μ-semiopen set in X is fuzzy μ-preopen in X.

Proof : Let λ be a fuzzy covered μ-semiopen set, and assume that $\beta \subset \lambda \subset c_{\mu}(\beta)$ for some μ-fuzzy open set β. Then, it follow that
$c_{\mu}(\lambda) = c_{\mu}(\beta)$.
Furthermore, there exists another fuzzy μ-open set γ such that
$\beta \subset \lambda \subset \gamma \subset c_{\mu}(\beta)$.
which implies that
$\lambda \subset i_{\mu}(\beta)) = i_{\mu}(c_{\mu}(\lambda))$.

Thus, λ is fuzzy μ-preopen.

Theorem 2.9. Let X be a fuzzy μ - precompact space. If every fuzzy μ - semiopen cover of X is a fuzzy μ - cover, then X is fuzzy S - Closed.

Proof. Suppose $\mathbb{Z}$ is fuzzy μ - semiopen cover of X. By Lemma 2.8, if follows that $\mathbb{Z}$ is a fuzzy μ-preopen cover of X. By Theorem 2.4, there exists a finite fuzzy μ - subfamily $\mathcal{M} \subseteq \mathbb{Z}$ such that the collection $\{i_{\mu}\left(c_{\mu}(\lambda)\right) | \lambda \in \mathcal{M}\}$ forms a cover of X.

Since each $\lambda \in \mathcal{M}$ satisfies

$$\lambda \subset i_{\mu}c_{\mu}(\lambda)) \subseteq c_{\mu}(\lambda)$$

if follow that the family $\{c_{\mu}(\lambda) | \lambda \in \mathcal{M}\}$ also covers X. Hence, X is fuzzy S - Closed.

A fuzzy subset λ of X is called fuzzy μ-precompact relative to X if every X_{μ}-PO cover of X with respect to λ admits a finite fuzzy μ-open supercover. Based on Theorem 2.4. this is equivalent to the condition that for any fuzzy μ-preopen cover $\mathbb{Z}$ of X for λ, there exists a finite fuzzy μ-subfamily $\mathcal{M} \subseteq \mathbb{Z}$ such that collection $\{i_{\mu}(c_{\mu}(\gamma) | \gamma \in \mathcal{M}\}$ covers λ.

Theorem 2.10. If every proper fuzzy regularly μ -closed subset of a μ-fuzzy space X is fuzzy μ-precompact relative to X, then X itself is fuzzy μ-precompact.

Proof : Let $\mathbb{Z} = \{\lambda_{\xi} | \xi \in \Delta\}$ be a fuzzy μ -preopen cover cover of X. Since $\mathbb{Z}$ is a fuzzy μ-cover of X, there exists at lease one $\lambda \in \mathbb{Z}$ such that $\lambda \neq 0_X$. By applying Lemma 2.1, the fuzzy interior of the fuzzy closure $i_{\mu}(c_{\mu}(\lambda))$ is fuzzy regularly μ -open, and thus the complement $X - i_{\mu}(c_{\mu}(\lambda))$ is fuzzy regulary μ -closed in X.

Given our hypothesis, this complement set must be fuzzy μ-precompact in X. Therefore, there exists a finite subcollection $\left\{\lambda_{\varepsilon_j} | \varepsilon_j \epsilon \Delta, j = 1,2, \ldots, n\right\}$ of satisfying

$$X - i_{\mu}\left(c_{\mu}(\lambda)\right) \subset \bigcup_{j=1}^{n} i_{\mu}\left(c_{\mu}\left(\lambda_{\varepsilon_j}\right)\right).$$

It follows that

$X \subset \left(\bigcup_{j=1}^{n} i_\mu\left(c_\mu\left(\lambda_{\varepsilon_j}\right)\right)\right) \cup i_\mu\left(c_\mu(\lambda)\right)$.

Hence, this finite union of fuzzy μ- preopen sets from $\mathfrak{L}$ covers X, and by Theorem 2.4, we conclude that X is fuzzy μ- precompact.

Definition-2.11. A fuzzy μ- filter base $\mathfrak{B}$ on a μ- fuzzy space X is said to fuzzy p_μ-converge to a fuzzy point $x_\alpha \in X$ if, for every fuzzy μ- preopen set λ in X containing x_α, there exists a member $\beta \in \mathfrak{B}$ such that β is contained in $i_\mu\left(c_\mu(\lambda)\right)$.

Definition-2.12. A fuzzy μ- filter base $\mathfrak{B}$ on a μ- fuzzy space X is said to fuzzy p_μ-accumulate at a fuzzy point $x_\alpha \in X$ if, for every fuzzy μ- preopen set λ in X containing x_α, it holds that $\beta \cap i_\mu\left(c_\mu(\lambda)\right) \neq 0_X$ for all $\beta \in \mathfrak{B}$.

Lemma-2.11. If a fuzzy base $\mathfrak{B}$ in X, p_μ-converges to a fuzzy point $x_\alpha \in X$, then the fuzzy μ- filter base is p_μ- accumulates to x_α.

Proof: By the fuzzy p_μ-convergence of $\mathfrak{B}$ to $x_\alpha \in X$, there exists an element $\beta \in \mathfrak{B}$ such that $\beta \cap i_\mu\left(c_\mu(\lambda)\right)$ for every fuzzy fuzzy μ-preopen set λ with $x_\alpha \in \lambda$.

Let $\gamma \in \mathfrak{B}$. Then, there exists an element $\omega \in \mathbb{Z}$ such that

$$\omega \subset \gamma \cap \beta \subset \beta \subset i_\mu\left(c_\mu(\lambda)\right).$$

This implies that

$$\omega \cap i_\mu\left(c_\mu(\lambda)\right) \neq 0_{X.}$$

Since $\omega \subset \gamma$, we also have

$$\gamma \cap i_\mu\left(c_\mu(\lambda)\right) \neq 0_{X.}$$

Thus, ω_μ –fuzzy p_μ- accumulates to $x_\alpha \in X$.

Lemma-2.12. Let $\mathbb{Z}$ be a fuzzy maximal μ- filter base in X. Then $\mathbb{Z}$ is fuzzy p_μ-converges to $x_\alpha \in X$ if and only if $\mathbb{Z}$ is fuzzy p_μ- accumulates to fuzzy $x_\alpha \in X$.

Proof: Since $\mathbb{Z}$ is a fuzzy μ- filter base, it follows from Lemma 2.11 that if $\mathbb{Z}$ fuzzy p_μ-converges to some point $x_\alpha \in X$, then $\mathbb{Z}$ fuzzy p_μ- accumulates to $x_\alpha \in X$.

Conversely, let $\mathbb{Z}$ be a fuzzy maximal μ- filter base, and assume that $\mathbb{Z}$ fuzzy p_μ-accumulates to some point $x_\alpha \in X$. If $\mathbb{Z}$ does not fuzzy p_μ-converge to x_α, then for each $\gamma \in \mathbb{Z}$, there exists a fuzzy μ- PO set λ containing x_α such that

$$\gamma \not\subset i_\mu\left(c_\mu(\lambda)\right),$$

i.e..,

$$\gamma \cap c_\mu\left(i_\mu(X-\lambda)\right) \neq 0_{X.}$$

Let

$$\mathcal{M} = \mathbb{Z} \cup \left\{\gamma \cap c_\mu\left(i_\mu(X-\lambda)\right) \mid \gamma \in \mathbb{Z}\right\}$$

Then, $\mathcal{M}$ is fuzzy μ- filter base that properly contains $\mathbb{Z}$. This contradicts the assumption that $\mathbb{Z}$ is a fuzzy maximal μ- filter base.

Theorem 2.13.

a) X is fuzzy μ- precompact.
b) Every fuzzy μ- filter base has a p_μ- accumulation point in X.
c) Every fuzzy maximal μ- filter base p_μ- converges to a point in X.

Proof: (a)⇒(b): Assume that there exists a fuzzy μ- filter base $\mathbb{Z} = \{\Pi_\alpha \mid \alpha \in \Delta\}$ in X and that $\mathbb{Z}$ does not fuzzy p_μ- accumulation in X. For each fuzzy point $x_\alpha \in X$, there exists a fuzzy μ- PO set Λ_{x_α} containing x_α, and a set $\Pi_{g(x_\alpha)} \in \mathbb{Z}$ such that

$$\Pi_{g(x_\alpha)} \cap i_\mu\left(c_\mu(\Lambda_{x_\alpha})\right) = 0.$$

Thus, the collection $\mathcal{L} = \left\{\Lambda_{x_\alpha} \mid x_\alpha \in X\right\}$ is a fuzzy μ- PO cover of X. By Theorem 2.4, there exists a finite fuzzy μ- subcollection of $\mathcal{L}$, say $\Lambda_{x_{\alpha_1}}, \Lambda_{x_{\alpha_2}}, \ldots., \Lambda_{x_{\alpha_n}}$, such that

$$\left\{i_\mu\left(c_\mu\left(\Lambda_{x_{\alpha_k}}\right)\right) \mid k \in \{1,2,\ldots\ldots,n\}\right\}$$

Covers X. Since $\mathbb{Z}$ is a fuzzy μ- filter base, there exists an element $\Pi_0 \in \mathbb{Z}$ such that

$$\Pi_0 \subset \bigcap_{k=1}^{n} \Pi_{g\left(x_{\alpha_k}\right)}.$$

This implies that

$$\Pi_0 \cap \left(\bigcup_{k=1}^{n} i_\mu\left(c_\mu\left(\Lambda_{x_{\alpha_k}}\right)\right)\right) = 0$$

Now, we have

$$\Pi_0 = \Pi_0 \cap X = \Pi_0 \cap \left(\cup_{k=1}^{n} i_\mu\,(c_\mu(\Lambda_{x_{\alpha_k}}))\right) = \cup_{k=1}^{n}\left(\Pi_0 \cap i_\mu(c_\mu(\Lambda_{x_{\alpha_k}}))\right) = 0.$$

This contradicts the fact that Π_0 is nontrivial, i.e., $\Pi_0 \neq 0$.

(b) ⇒ (c) : Let $\mathbb{Z}$ be a fuzzy maximal μ- filter base in X. By assumption (ii), the fuzzy base $\mathbb{Z}\, p_{\mu} -$ accumulates to some point $x_0 \in X$. Since $\mathbb{Z}$ is a fuzzy maximal μ- filter base in X, it follows from Lemma 2.11 that $\mathbb{Z}$ p_{u} −converges to $x_0 \in X$.

(c) ⇒ (a): Let $\mathcal{L} = \{\Lambda_{\alpha} \mid \alpha \in \Delta\}$ be a fuzzy μ-preopen cover of X. Assume, for contradiction, that X is not fuzzy μ-precompact. Then, for each finite fuzzy μ-subcollection Δ_0 of Δ, we have

$$\bigcup_{\alpha\in\Delta_0} i_{\mu}\left(c_{\mu}(\Lambda_{\alpha})\right) \neq X,$$

which implies

$$\cap_{\alpha\in\Delta_0} c_{\mu}\left(i_{\mu}(X-\Lambda_{\alpha})\right) \not\equiv 0.$$

Let

$$\Pi_{\Delta_0} = \bigcap_{\alpha\in\Delta_0} c_{\mu}\left(i_{\mu}(X-\Lambda_{\alpha})\right).$$

Let Λ be the collection of all finite fuzzy μ- subcollection of Δ. and define

$$\mathbb{Z} = \{\Pi_{\lambda} | \lambda \in \Lambda\},$$

where each Π_{λ} corresponds to the set Π_{Δ_0} for a finite μ-subcollection of Δ.We observe that $\mathbb{Z}$ is a fuzzy μ- filter base on X, so there exists a fuzzy maximal μ-filter base $\mathcal{M}$ containing $\mathbb{Z}$. By part (c), $\mathcal{M}\, p_{\mu}$-converges to some point $x_0 \in X$ and $\mathcal{M}$ μ-fuzzy p_{μ}-accumulates to $x_0 \in X$ by Lemma 2.12. Since $\mathcal{L}$ is a fuzzy μ- cover of X, there exists a $\Lambda_0 \in \mathcal{L}$ such that $x_0 \in \Lambda_0$.

Thus, by construction,

$$c_{\mu}\left(i_{\mu}(X-\Lambda_0)\right) \in \mathcal{M}$$

Since $\mathcal{M}\, p_{\mu}$-accumulates to x_0 and $x_0 \in \Lambda_0$, we see that

$$\Pi \cap i_{\mu}\left(c_{\mu}(\Lambda_0)\right) \not\equiv 0$$

For each $\Pi \in \mathcal{M}$. In particular,

$$c_{\mu}\left(i_{\mu}(X-\Lambda_0)\right) \cap i_{\mu}\left(c_{\mu}(\Lambda_0)\right) \not\equiv 0$$

Which contradicts the fact that

$$c_{\mu}\left(i_{\mu}(X-\Lambda_0)\right) \cap i_{\mu}\left(c_{\mu}(\Lambda_0)\right) \equiv 0.$$

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