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Impact of Prey Harvesting on the Stability of a Three-Species Food Chain Model with a Generalist Predator

S. Ganga[1*] and S. Vijaya[2]

[1*]Research Scholar, Department of Mathematics, Annamalai University, Annamalai Nagar,
Chidambaram, Tamil Nadu, India - 608 002.
Email: [*1] gangasanthru@gmail.com

[2]Professor, Department of Mathematics, Annamalai University, Annamalai Nagar,
Chidambaram, Tamil Nadu, India - 608 002.
Email: [2]havenksho@gmail.com

Abstract

In this study, which involves a prey, an intermediate specialist predator, and a top predator exhibiting generalist behavior, we examine the stability of a three-species food chain model. Harvesting is applied to the prey. The primary objective is to investigate how varying the harvesting rate influences the stability of the system, particularly the coexistence of all three species. We analyze the positivity, boundedness, equilibrium points, and stability of the model, and employ numerical simulations to support the theoretical analysis and explore how different harvesting intensities affect species persistence.

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1 Introduction

Predator-prey interactions are crucial for ecological balance and are common throughout the biosphere. They also impact human life, especially through the use of renewable resources such as fisheries and forests. Lotka [10] and Volterra [16] were the first to model these interactions mathematically. While harvesting is a form of commercial exploitation, overharvesting can disrupt or collapse ecosystems. Brauer and Soudack [3] studied a prey-predator model with constant-rate predator harvesting. The Leslie-Gower prey-predator model with linear harvesting in both species was analyzed by Beddington and May [1]. A ratio-dependent prey-predator model with constant-rate prey harvesting was investigated by Xiao and Jennings [14], while the same model with constant predator harvesting was examined by Xiao et al.[15] , where different dynamics were observed. A ratio-dependent prey-predator model incorporating both linear and nonlinear prey harvesting was explored by Lenzini and Rebaza [9]. A Leslie-Gower predator-prey model with constant-rate prey harvesting was proposed by Zhu and Lan [17], and saddle-node as well as subcritical or supercritical Hopf bifurcations were discussed. Gong and Huang [5] examined the Bogdanov-Takens bifurcation for this model.

In 2012, a stage-structured prey-predator model with predator harvesting was studied by Chakraborty et al. [4]. The impact of both phytoplankton and zooplankton population harvesting was analyzed by Lv et al. [11]. Subsequently, a modified Leslie-Gower type prey-predator model with Michaelis-Menten type prey harvesting was investigated by Gupta and Chandra [7]. A Leslie-Gower prey-predator model with nonlinear prey harvesting was analyzed by Gupta et al.[6]. Criteria for the existence of a bionomic equilibrium were derived, and Pontryagin's maximum principle was applied to determine the optimal harvesting policy. A Leslie-Gower prey-predator model with Michaelis-Menten type predator harvesting was investigated by Singh et al. [13].

In this study, which involves a prey, an intermediate specialist predator, and a top predator exhibiting generalist behavior, we examine the stability of a three-species food chain model. Harvesting is applied to the prey. The primary objective is to investigate how varying the harvesting rate influences the stability of the system, particularly the coexistence of all three species. We analyze the boundedness, positivity, equilibrium points, and stability of the model, and employ numerical simulations to support the theoretical analysis and explore how different harvesting intensities affect species persistence.

This paper follows the structure below. The mathematical formulation of a three-species population model is presented in Segment 2. The boundedness and positivity of the system are examined in Segment 3. In Segment 4, the local stability analysis of the nonlinear system is discussed, followed by the global stability assessment in Segment 5. Segment 6 provides numerical simulations, and the study is concluded in the final segment.

2 Model

We present a mathematical formulation involving prey, an intermediate specialist predator, and a top predator exhibiting generalist behavior in a three-species food chain model. Let $M_1(t), M_2(t)$, and $M_3(t)$ represent their densities at any time $t > 0$. The intermediate predator's consumption of prey follows a Holling Type I response, expressed as $b_1 M_1 M_2$. The prey M_1, serves as the only food source for the intermediate predator, which is a specialist. The top predator's consumption of the intermediate predator follows a Holling Type II response, expressed as $\frac{b_2 M_2 M_3}{1+hM_2}$. If the intermediate predator disappears, the top predator, being a generalist, can survive on alternative food sources. Additionally, the model incorporates harvesting of the prey. The mathematical formulation is as follows:

$$\begin{cases} \dfrac{dM_1}{dt} = rM_1\left(1-\dfrac{M_1}{k}\right) - b_1M_1M_2 - \delta_0M_1, \\ \dfrac{dM_2}{dt} = \beta_1b_1M_1M_2 - \dfrac{b_2M_2M_3}{1+hM_2} - \gamma_1M_2, \\ \dfrac{dM_3}{dt} = \beta_2\dfrac{b_2M_2M_3}{1+hM_2} + \dfrac{\eta M_3}{1+\eta_0M_3} - \gamma_2M_3. \end{cases} \tag{1}$$

Given $M_1(0) \geq 0$, $M_2(0) \geq 0$ and $M_3(0) \geq 0$ as the initial conditions.

Each parameter is assumed to have a positive value. Observe that the term $rM_1\left(1-\frac{M_1}{k}\right)$ denotes the logistic growth of prey, while the generalist predator's growth, characterized by a Beverton-Holt-like function $\frac{\eta M_3}{1+\eta_0M_3}$, relies on the presence of supplementary food sources. Moreover, we consider the scenario $\eta > \gamma_2$, as it has been shown to be crucial for the persistence of the top predator when the intermediate predator does not exist [12].

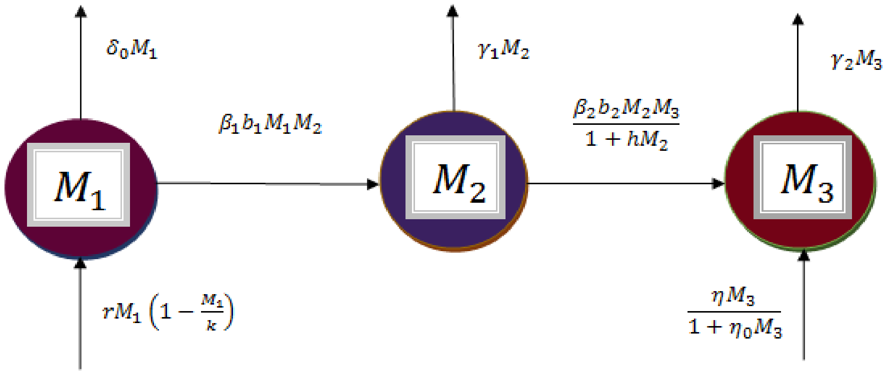

Figure1: Structural diagram of system(1)

TABLE 1: Description of Parameters

Parameter	**Description**
r	Natural growth rate of the prey
k	Carrying capacity of the prey
b_1	Predation rate coefficient of the intermediate predator on the prey
b_2	Predation rate coefficient of the top predator on the intermediate predator
β_1	Growth efficiency of the intermediate predator from prey consumption
β_2	Growth efficiency of the top predator from intermediate predator consumption
h	Handling time
η	Reproduction rate per individual of the top predator
η_0	Density-dependent power of the top predator
γ_1	Natural death rate of the intermediate predator
γ_2	Natural death rate of the top predator
δ_0	Prey's harvesting rate

3 Examination of the system's positivity and boundedness

Theorem 3.1. System (1) yields solutions that are positive for all t ≥ 0.

Proof. Integrating each equation in model(1) yields

$$M_1(t) = M_1(0)\exp\left\{\int [r\left(1-\frac{M_1}{k}\right) - b_1M_2 - \delta_0]dt\right\},$$
$$M_2(t) = M_2(0)\exp\left\{\int [\beta_1 b_1 M_1 - \frac{b_2M_3}{1+hM_2} - \gamma_1]dt\right\},$$
$$M_3(t) = M_3(0)\exp\left\{\int [\beta_2 \frac{b_2M_2}{1+hM_2} + \frac{\eta}{1+\eta_0 M_3} - \gamma_2]dt\right\}.$$

Since, $M_1(0) \geq 0, M_2(0) \geq 0$, and $M_3(0) \geq 0$, we obtain $M_1(t) \geq 0, M_2(t) \geq 0$, and $M_3(t) \geq 0$ for all $t \geq 0$. Therefore, the system (1) yields solutions that are positive for all $t \geq 0$.

Theorem 3.2. The trajectories of system (1) are uniformly bounded under the initial conditions

Proof. let the operation $V = M_1 + M_2 + M_3$ be defined, and upon differentiating V with respect to t, the result is

$$\frac{dV}{dt}=\frac{dM_1}{dt}+\frac{dM_2}{dt}+\frac{dM_3}{dt}\frac{dV}{dt}=rM_1\left(1-\frac{M_1}{k}\right)-\delta_0M_1-\gamma_1M_2+\frac{\eta M_3}{1+\eta_0M_3}-\gamma_2M_3.$$

For an arbitrary positive real number $\breve{\varpi}>0$,

$$\frac{dV}{dt}+\breve{\varpi}V=\left[r\left(1-\frac{M_1}{k}\right)-\delta_0+\breve{\varpi}\right]M_1+(\breve{\varpi}-\gamma_1)M_2+\left[\frac{\eta}{1+\eta_0M_3}-\gamma_2+\breve{\varpi}\right]M_3.$$

If $\breve{\varpi}\leq\gamma_1$, then $\frac{dV}{dt}+\breve{\varpi}V\leq\frac{k(r+\breve{\varpi}-\delta_0)}{r}+\frac{\eta-\gamma_2+\breve{\varpi}}{\eta_0(\gamma_2-\breve{\varpi})}\leq\breve{\Psi}$,

where $\breve{\Psi}=\frac{k(r+\breve{\varpi}-\delta_0)}{r}+\frac{\eta-\gamma_2+\breve{\varpi}}{\eta_0(\gamma_2-\breve{\varpi})}$.

From the theorem of differential inequalities[2], the result follows:

$$0\leq V(M_1,M_2,M_3)\leq\left(\frac{\breve{\Psi}}{\breve{\varpi}}\right)(1-e^{-\breve{\varpi}t})+V(M_1(0),M_2(0),M_3(0))e^{-\breve{\varpi}t}.$$

When $t\to\infty, 0\leq V(M_1,M_2,M_3)\leq\frac{\breve{\Psi}}{\breve{\varpi}}$. Therefore, every solution of (1) that begins in R_+^3 is confined with the region

$$B=\{(M_1,M_2,M_3)\in R_+^3: 0\leq V(M_1,M_2,M_3)\leq\frac{\breve{\Psi}}{\breve{\varpi}}+\epsilon, \text{for any } \epsilon>0\}.$$

Thus, the trajectories of system (1) remain uniformly bounded when $M_1(0)\geq 0, M_2(0)\geq 0$, and $M_3(0)\geq 0$.

4 Examination of local stability in a nonlinear system

4.1 Equilibrium points

The equilibrium points of the model are essential in analyzing the local stability features of the predator-prey model. Model (1) contains six equilibrium points.

TABLE 2: Conditions for the Existence of Equilibrium Points and Their Biological Interpretations

Equilibrium point	Existence Condition	Description
$E_0(0,0,0)$	Exists	Trivial

$E_1\left(\frac{k(r-\delta_0)}{r},0,0\right)$	Exists if $r>\delta_0$	Only the prey persists
$E_2\left(0,0,\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right)$	Exists	Only the top predator persists
$E_3(\widetilde{M}_1,\widetilde{M}_2,0)$, where $\widetilde{M}_1=\frac{\gamma_1}{\beta_1 b_1}$, $\widetilde{M}_2=\frac{1}{b_1}\left[r-\frac{r\gamma_1}{k\beta_1 b_1}-\delta_0\right]$	Exists if $r>\frac{r\gamma_1}{k\beta_1 b_1}+\delta_0$	Only the prey and Intermediate predator persist
$E_4\left(\frac{k(r-\delta_0)}{r},0,\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right)$	Exists $r>\delta_0$	Only the prey and top predator persist
$E_5({M_1}^*,{M_2}^*,{M_3}^*)$	Exists if $M_1^*>0$, $M_2^*>0, M_3^*>0$	Coexistence

4.2 Examination of the characteristics of equilibrium points and their local stability

At the equilibrium point $E(M_1,M_2,M_3)$, the variational matrix of system(1) is

$$J=\begin{bmatrix}\hat{c}_{11} & \hat{c}_{12} & 0\\ \hat{c}_{21} & \hat{c}_{22} & \hat{c}_{23}\\ 0 & \hat{c}_{32} & \hat{c}_{33}\end{bmatrix},$$

where

$$\hat{c}_{11}=r-\frac{2rM_1}{k}-b_1M_2-\delta_0,\qquad \hat{c}_{12}=-b_1M_1\,,\quad \hat{c}_{21}=\beta_1b_1M_2,$$
$$\hat{c}_{22}=\beta_1b_1M_1-\frac{b_2M_3}{(1+hM_2)^2}-\gamma_1,\quad \hat{c}_{23}=-\frac{b_2M_2}{1+hM_2}\,,\quad \hat{c}_{32}=\frac{\beta_2b_2M_3}{(1+hM_2)^2}\,,$$
$$\hat{c}_{33}=\frac{\beta_2b_2M_2}{1+hM_2}+\frac{\eta}{(1+\eta_0M_3)^2}-\gamma_2$$

Theorem 4.2.1. The equilibrium points E_0, E_1, and E_3 are always unstable.

Proof. The variational matrix J evaluated at E_0 has eigenvalues $\lambda_1=r-\delta_0, \lambda_2=-\gamma_1, \lambda_3=\eta-\gamma_2$. Since $\eta>\gamma_2$, at least one eigenvalue is positive, implying that E_0 is unstable. The variational matrix J evaluated at E_1 has eigenvalues $\lambda_1=-r+\delta_0, \lambda_2=\frac{\beta_1b_1k(r-\delta_0)}{r}-\gamma_1$,

$\lambda_3 = \eta - \gamma_2$. Since $\eta > \gamma_2$, at least one eigenvalue is positive, implying that E_1 is unstable. The variational matrix J evaluated at E_3 has the following eigenvalues:

$$\lambda_1 = \beta_2 A_2 + \eta - \gamma_2 > 0,$$

$$\lambda_2 = \frac{-\frac{r\gamma_1}{\beta_1 b_1 k} + \sqrt{\left(\frac{r\gamma_1}{\beta_1 b_1 k}\right)^2 - 4A_1\frac{\gamma_1}{\beta_1}}}{2},$$

$$\lambda_3 = \frac{-\frac{r\gamma_1}{\beta_1 b_1 k} - \sqrt{\left(\frac{r\gamma_1}{\beta_1 b_1 k}\right)^2 - 4A_1\frac{\gamma_1}{\beta_1}}}{2}$$

Where $A_1 = \beta_1 b_1 \widetilde{M}_2$,

$A_2 = \frac{b_2\widetilde{M}_2}{1+h\widetilde{M}_2}$.

Since $\lambda_1 > 0$, at least one eigenvalue is positive, implying that E_3 is unstable.

Theorem 4.2.2. The equilibrium point E_2 exhibits local asymptotic stability if $r < \delta_0$

Proof. The variational matrix J evaluated at E_2 has the following eigenvalues: $\lambda_1 = r - \delta_0, \lambda_2 = -b_2\left(\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right) - \gamma_1$, and $\lambda_3 = -\frac{\gamma_2}{\eta}[\eta - \gamma_2]$. Since $\eta > \gamma_2$, we have $\lambda_2 < 0$ and $\lambda_3 < 0$. Therefore, the equilibrium point E_2 exhibits local asymptotic stability if $r < \delta_0$.

Theorem 4.2.3. The equilibrium point E_4 exhibits local asymptotic stability if $r > \delta_0$ and $\frac{\beta_1 b_1 k(r-\delta_0)}{r} < b_2\left(\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right) + \gamma_1$.

Proof. The variational matrix J evaluated at E_4 has the following eigenvalues: $\lambda_1 = -r + \delta_0, \lambda_2 = \frac{\beta_1 b_1 k(r-\delta_0)}{r} - b_2\left(\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right) - \gamma_1$, $\lambda_3 = -\frac{\gamma_2}{\eta}[\eta - \gamma_2]$. Since $\eta > \gamma_2$, we have $\lambda_3 < 0$. Therefore, the equilibrium point E_4 exhibits local asymptotic stability if $r > \delta_0$ and $\frac{\beta_1 b_1 k(r-\delta_0)}{r} < b_2\left(\frac{\eta-\gamma_2}{\eta_0\gamma_2}\right) + \gamma_1$.

Theorem 4.2.4. The internal equilibrium point E_5 attains local asymptotic stability if and only if $\hat{C}_1 > 0, \hat{C}_3 > 0$ and $\hat{C}_1\hat{C}_2 - \hat{C}_3 > 0$ are satisfied.

Proof. Given system (1) at E_5, the variational matrix is

$$J^* = \begin{bmatrix} \hat{c}_{11}{}^* & \hat{c}_{12}{}^* & 0 \\ \hat{c}_{21}{}^* & \hat{c}_{22}{}^* & \hat{c}_{23}{}^* \\ 0 & \hat{c}_{32}{}^* & \hat{c}_{33}{}^* \end{bmatrix},$$

Where

$$\hat{c}_{11}{}^{*} = r - \frac{2rM_1{}^{*}}{k} - b_1M_2{}^{*} - \delta_0, \qquad \hat{c}_{12}{}^{*} = -b_1M_1{}^{*},$$

$$\hat{c}_{21}{}^{*} = \beta_1b_1M_2{}^{*}, \qquad \hat{c}_{22}{}^{*} = \beta_1b_1M_1{}^{*} - \frac{b_2M_3{}^{*}}{(1+hM_2{}^{*})^2} - \gamma_1,$$

$$\hat{c}_{23}{}^{*} = -\frac{b_2M_2{}^{*}}{1+hM_2{}^{*}}, \qquad \hat{c}_{32}{}^{*} = \frac{\beta_2b_2M_3{}^{*}}{(1+hM_2{}^{*})^2},$$

$$\hat{c}_{33}{}^{*} = \frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{(1+\eta_0M_3{}^{*})^2} - \gamma_2.$$

The variational matrix J^* has a characteristic equation of

$$\lambda^3 + \hat{C}_1\lambda^2 + \hat{C}_2\lambda + \hat{C}_3 = 0,$$

where

$$\hat{C}_1 = -(\hat{c}_{11}{}^{*} + \hat{c}_{22}{}^{*} + \hat{c}_{33}{}^{*})$$

$$= \frac{2rM_1{}^{*}}{k} + b_1M_2{}^{*} + \delta_0 - r - \beta_1b_1M_1{}^{*} + \frac{b_2M_3{}^{*}}{(1+hM_2{}^{*})^2} + \gamma_1 - \frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} - \frac{\eta}{(1+\eta_0M_3{}^{*})^2} + \gamma_2.$$

$$\hat{C}_2 = \hat{c}_{22}{}^{*}\ \hat{c}_{33}{}^{*} - \hat{c}_{23}{}^{*}\ \hat{c}_{32}{}^{*} + \hat{c}_{11}{}^{*}\ \hat{c}_{33}{}^{*} + \hat{c}_{11}{}^{*}\ \hat{c}_{22}{}^{*} - \hat{c}_{12}{}^{*}\ \hat{c}_{21}{}^{*}$$

$$= \left[\left(\beta_1b_1M_1{}^{*} - \frac{b_2M_3{}^{*}}{(1+hM_2{}^{*})^2} - \gamma_1\right)\left(\frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{(1+\eta_0M_3{}^{*})^2} - \gamma_2\frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{(1+\eta_0M_3{}^{*})^2} - \gamma_2\right)\right] + \left[\frac{\beta_2b_2{}^{2}M_2{}^{*}M_3{}^{*}}{(1+hM_2{}^{*})^3}\right] + \left[\left(r - \frac{2rM_1{}^{*}}{k} - b_1M_2{}^{*} - \delta_0\right)\left(\frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{(1+\eta_0M_3{}^{*})^2} - \gamma_2\right)\right] + \left[\left(r - \frac{2rM_1{}^{*}}{k} - b_1M_2{}^{*} - \delta_0\right)\left(\beta_1b_1M_1{}^{*} - \frac{b_2M_3{}^{*}}{(1+hM_2{}^{*})^2} - \gamma_1\right)\right] + \left[\beta_1b_1{}^{2}M_1{}^{*}M_2{}^{*}\right].$$

$$\hat{C}_3 = \hat{c}_{12}{}^{*}\ \hat{c}_{21}{}^{*}\ \hat{c}_{33}{}^{*} + \hat{c}_{11}{}^{*}\ \hat{c}_{23}{}^{*}\ \hat{c}_{32}{}^{*} - \hat{c}_{11}{}^{*}\ \hat{c}_{22}{}^{*}\ \hat{c}_{33}{}^{*}$$

$$= \left[\left(r - \frac{2rM_1{}^{*}}{k} - b_1M_2{}^{*} - \delta_0\right)\left(-\frac{\beta_2b_2{}^{2}M_2{}^{*}M_3{}^{*}}{\left(1+hM_2{}^{*}\right)^3}\right)\right] + \left[\left(-\beta_1b_1{}^{2}M_1{}^{*}M_2{}^{*}\right)\left(\frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{\left(1+\eta_0M_3{}^{*}\right)^2} - \gamma_2\right)\right] - \left[\left(r - \frac{2rM_1{}^{*}}{k} - b_1M_2{}^{*} - \delta_0\right)\left(\beta_1b_1M_1{}^{*} - \frac{b_2M_3{}^{*}}{\left(1+hM_2{}^{*}\right)^2} - \gamma_1\right)\left(\frac{\beta_2b_2M_2{}^{*}}{1+hM_2{}^{*}} + \frac{\eta}{\left(1+\eta_0M_3{}^{*}\right)^2} - \gamma_2\right)\right].$$

Therefore, the internal equilibrium point E_5 attains local asymptotic stability if and only if $\hat{C}_1 > 0, \hat{C}_3 > 0$ and $\hat{C}_1\hat{C}_2 - \hat{C}_3 > 0$ hold, based on the Routh-Hurwitz criteria.

5 Global stability analysis

Theorem 5.1. The equilibrium point E_4 exhibits global asymptotic stability.

Proof. Let $H(M_1, M_3) = \frac{1}{M_1 M_3}$. It is evident that $H(M_1, M_3)$ is positive when $M_1 > 0, M_3 > 0$. Let $f_1(M_1, M_3) = rM_1\left(1 - \frac{M_1}{k}\right) - \delta_0 M_1$ and $f_2(M_1, M_3) = \frac{\eta M_3}{1+\eta_0 M_3} - \gamma_2 M_3$. Then $\Delta(M_1, M_3) = \frac{\partial}{\partial M_1}(f_1 H) + \frac{\partial}{\partial M_3}(f_2 H) = -\frac{r}{kM_3} - \frac{\eta\eta_0}{M_1(1+\eta_0 M_3)^2} < 0$. According to the Bendixson-Dulac criterion[8], $\Delta(M_1, M_3)$ retains a constant sign and is not identically zero throughout the positive quadrant of the $M_1 - M_3$ plane.

Therefore, the equilibrium point E_4 exhibits global asymptotic stability.

Theorem 5.2. The internal equilibrium point E_5 exhibits global asymptotic stability if the following conditions are satisfied:

(i) $M_1 > {M_1}^*, M_2 > {M_2}^*, M_3 > {M_3}^*$,

(ii) ${M_2}^* M_3 > M_2 {M_3}^*$.

Proof. The Lyapunov stability theorem is used to derive the proof, as it provides a fundamental criterion for stability.

Now, let us define a positive definite function W as follows:

$$W = e_1\left[(M_1 - {M_1}^*) - {M_1}^* \, ln\left(\frac{M_1}{{M_1}^*}\right)\right] + e_2\left[(M_2 - {M_2}^*) - {M_2}^* \, ln\left(\frac{M_2}{{M_2}^*}\right)\right] + e_3\left[(M_3 - {M_3}^*) - {M_3}^* \, ln\left(\frac{M_3}{{M_3}^*}\right)\right] \quad (2)$$

Here, e_1, e_2 and e_3 are positive constants that will be determined later.

Differentiate (2) with respect to t, we get

$$\frac{dW}{dt} = e_1\left[\frac{M_1 - {M_1}^*}{M_1}\right]\frac{dM_1}{dt} + e_2\left[\frac{M_2 - {M_2}^*}{M_2}\right]\frac{dM_2}{dt} + e_3\left[\frac{M_3 - {M_3}^*}{M_3}\right]\frac{dM_3}{dt}$$

$$= e_1\left[r\left(1 - \frac{M_1}{k}\right) - b_1 M_2 - \delta_0\right](M_1 - {M_1}^*) + e_2\left[\beta_1 b_1 M_1 - \frac{b_2 M_3}{1 + hM_2} - \gamma_1\right](M_2 - {M_2}^*)$$

$$+ e_2\left[\beta_2 \frac{b_2 M_2}{1 + hM_2} + \frac{\eta}{1 + \eta_0 M_3} - \gamma_2\right](M_3 - {M_3}^*)$$

$$= -\frac{e_1 r}{k}(M_1 - {M_1}^*)^2 - b_1[e_1 - e_2\beta_1](M_1 - {M_1}^*)(M_2 - {M_2}^*) - b_1\left[\frac{e_2 - e_3\beta_2}{(1+hM_2)(1+h{M_2}^*)}\right](M_2 - {M_2}^*)(M_3 - {M_3}^*) -$$

$$e_2b_2h\left[\frac{{M_2}^*M_3-M_2{M_3}^*}{(1+hM_2)(1+h{M_2}^*)}\right](M_2-{M_2}^*)-e_3\left[\frac{\eta\eta_0}{(1+\eta_0M_3)(1+\eta_0{M_3}^*)}\right](M_3-{M_3}^*)^2$$

By selecting positive constant values such that $e_1=\beta_1, e_2=1, e_3=\frac{1}{\beta_2}$, we ensure that, under the given conditions (i)-(ii), $\frac{dW}{dt}$ is negative definite, establishing W as a Lyapunov function for E_5. Therefore, the internal equilibrium point E_5 exhibits global asymptotic stability.

6 Numerical simulation

To examine the impact of harvesting on system dynamics, numerical simulations are conducted by fixing the parameter values as follows: $b_1=1, h=0.1, k=2, r=1, \beta_1=1.6, \gamma_1=0.5, b_2=0.5, \beta_2=1, \gamma_2=1, \eta=2, \eta_0=1$. Only the harvesting rate δ_0 is varied.

Figure 2 shows the bifurcation diagram with respect to δ_0, indicating that an increase in prey harvesting reduces the populations of all species.

In Figure 2, a stable coexistence equilibrium point E_5 is observed in the range $0\leq\delta_0<0.688$, where all three species survive with positive densities. This equilibrium point is locally asymptotically stable within this range. For example, when $\delta_0=0.4<0.688$, the conditions of Theorem 4.2.4 are satisfied: $\hat{C}_1=0.8287>0, \hat{C}_3=0.1613>0$, $\hat{C}_1\hat{C}_2-\hat{C}_3=0.2648>0$. Therefore, based on Theorem 4.2.4, the equilibrium point $E_5(0.7017,0.2493,1.277)$ is locally asymptotically stable with initial value (1.25,0.36,1.5) (see Figure 3(a)), indicating that the population remains stable at $E_5(0.7017,0.2493,1.277)$, which allows for the construction of a phase diagram (see Figure 3(b)).

As h_1 increases to the range $0.688\leq\delta_0<1$, no interior equilibrium point E_5 exists; the intermediate predator goes extinct, and the system moves toward the boundary equilibrium point E_4, which exhibits local asymptotic stability and is characterized by the survival of only the prey and the top predator. For instance, when $0.688<\delta_0=0.75<1$, the eigenvalues of $E_4(0.5,0,1)$ are: $\lambda_1=-0.25$, $\lambda_1=-0.2$ and $\lambda_3=-0.5$,showing that $E_4(0.5,0,1)$ is locally asymptotically stable with initial value (1.25,0.36,1.5) (see Figure 4(a)), allowing the construction of a phase diagram (see Figure 4(b)).

When $\delta_0\geq1$, neither an interior nor a boundary equilibrium point exists. The harvesting pressure on the prey equals or exceeds its natural growth rate, leading to the extinction of the prey population. Since the intermediate predator is already extinct, the system reaches the equilibrium point E_2, where only the top predator persists, surviving solely on its alternative food resources. When $\delta_0=1$, the eigenvalues of $E_2(0,0,1)$ are: $\lambda_1=0, \lambda_2=-1$, and $\lambda_3 = -0.5$, indicating a non-hyperbolic equilibrium that is marginally stable. When $\delta_0 > 1$, the equilibrium point becomes locally asymptotically stable. For example, when $\delta_0 = 1.5 > 1$, the eigenvalues of $E_2(0,0,1)$ are: $\lambda_1 = -0.5, \lambda_2 = -1$, and $\lambda_3 = -0.5$, indicating that $E_2(0,0,1)$ is locally asymptotically stable with initial value (0.4,0.36,1.5) (see Figure 5(a)), which allows for the construction of a phase diagram (see Figure 5(b)).

Thus, the transition from the coexistence equilibrium $(E_5) \rightarrow$ boundary equilibrium $(E_4) \rightarrow$ top predator-only equilibrium (E_2) as δ_0 increases highlights the sensitive dependence of the food chain on harvesting intensity and reveals the potential for species extinction when the harvesting rate exceeds certain critical thresholds.

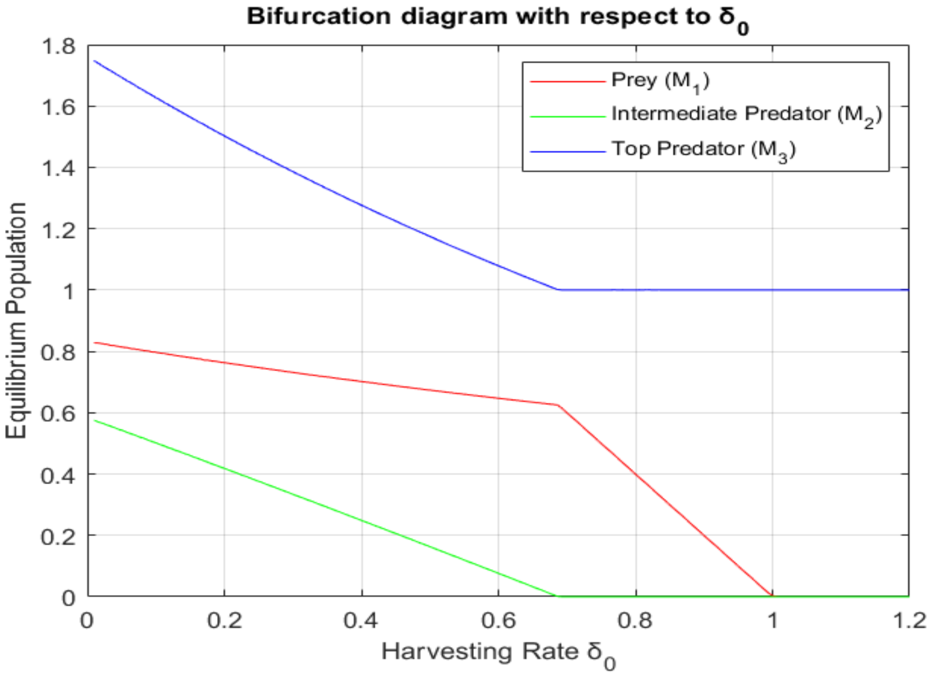

Figure 2: Bifurcation diagram of system(1) with respect to δ_0

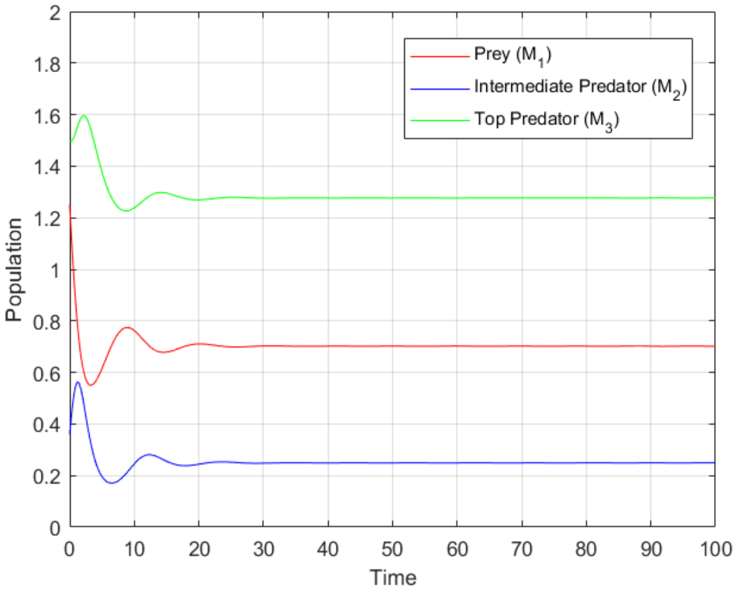

Figure 3 (a): Time series for $\delta_0 = 0.4$, showing stable coexistence at $E_5(0.7017, 0.2493, 1.277)$.

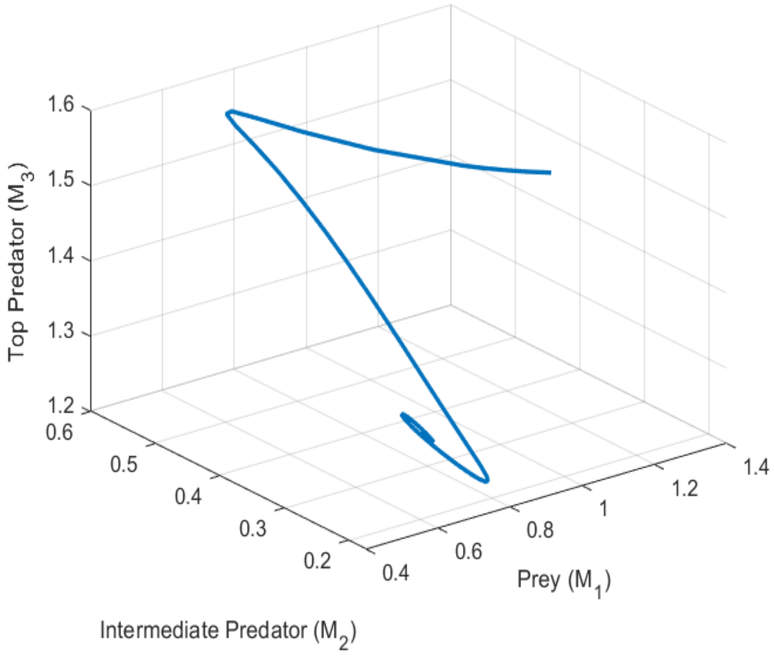

Figure 3 (b): Phase diagram

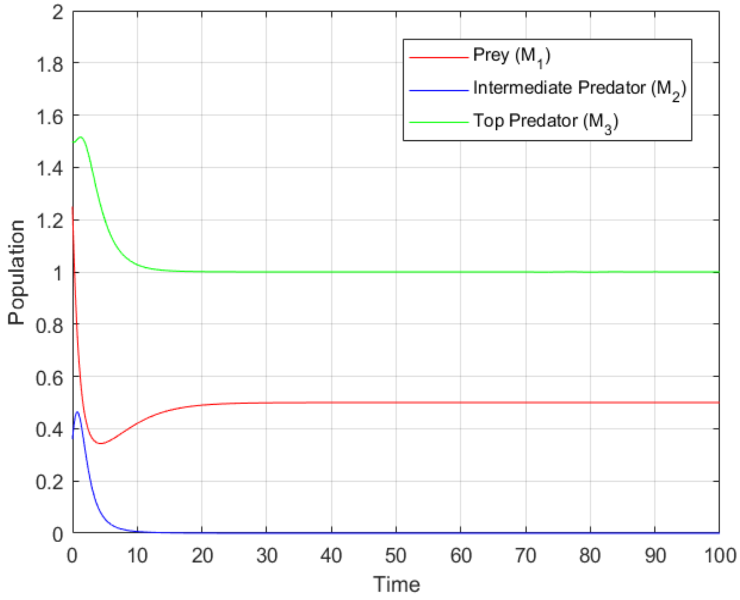

Figure 4 (a): Time series for $\delta_0 = 0.75$, showing stability at the boundary equilibrium point $E_4(0.5,0,1)$.

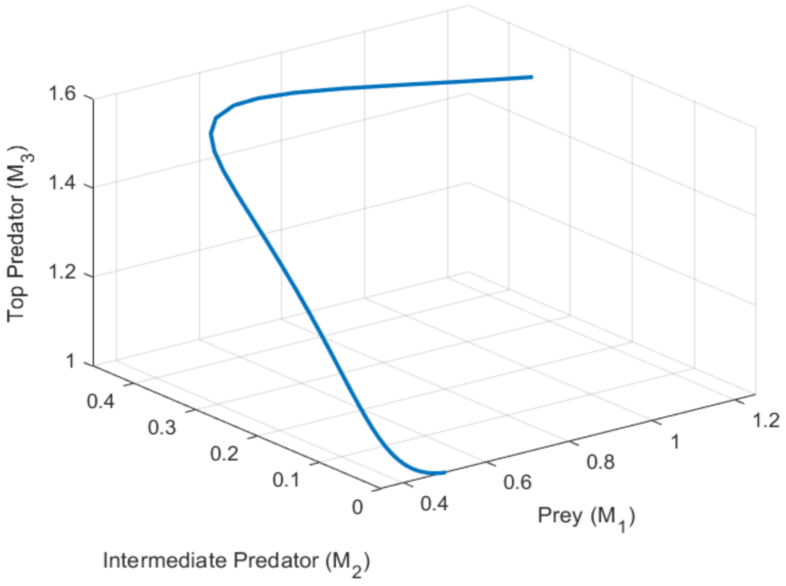

Figure 4 (b): Phase diagram.

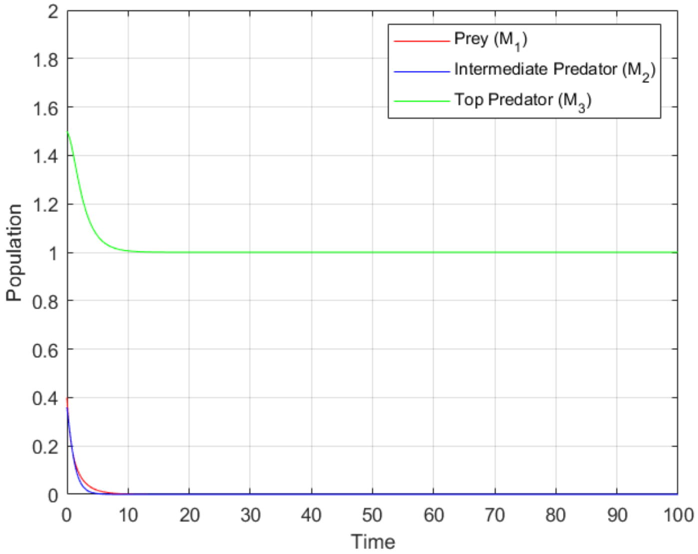

Figure 5(a): Time series for $\delta_0 = 1.5$, showing stability at the equilibrium point $E_2(0,0,1)$.

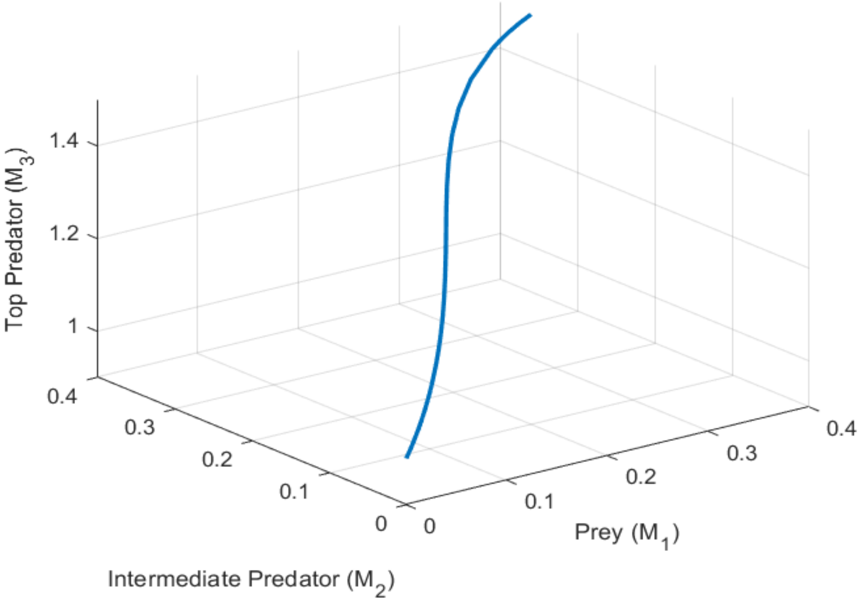

Figure 5(b): Phase diagram.

7 Conclusion

In this study, the stability analysis of a three-species food chain model incorporating a top predator (generalist) was conducted, with a focus on the impact of prey harvesting on the

system. First, the boundedness and positivity conditions of the model were established. Local stability analysis showed that the possible equilibrium points satisfied the criteria for local asymptotic stability. Global stability analysis further demonstrated that these equilibrium points were globally stable under certain conditions.
Numerical simulations were conducted to explore the consequences of varying the prey harvesting rate. The results demonstrate that harvesting can lead to significant changes in population dynamics, including species extinction and shifts in the equilibrium structure. As the prey harvesting rate increases, the system transitions from stable coexistence of all species to partial survival of only the prey and top predator (with the intermediate predator going extinct), and finally to the survival of only the top predator. The top predator does not go extinct in this system, as it continues to survive by consuming alternative food resources.
Overall, the study provides valuable insights into how harvesting pressure influences food chain stability and offers a theoretical foundation for sustainable resource management.

References

[1] J. R. Beddington and R. M. May, Maximum sustainable yields in systems subject to harvesting at more than one trophic level, *Mathematical Biosciences,* **51** (1980), 261–281 .

[2] G. Birkhoff and G. C. Rota, Ordinary differential equations, *Blaisdell Publishing Company, Ginn and Company, Waltham, Massachusetts;Toronto, Ontario;London* , (1969).

[3] F. Brauer and A.C. Soudack, Stability regions and transition phenomena for harvested predator-prey system, *Journal of Mathematical Biology,* **7** (1979), 319–337.

[4] K. Chakraborty, S. Jana and T. K. Kar, Global dynamics and bifurcation in a stage structured prey-predator fishery model with harvesting, *Applied Mathematics and Computation,* **218** (18)(2012), 9271–9290.

[5] Y. Gong and J. Huang, Bogdanov-Takens bifurcation in a Leslie-Gower predator-prey model with prey harvesting, *Acta Mathematicae Applicatae Sinica, English Series,* **30**(1) (2014), 239–244.

[6] R. P. Gupta, M. Banarjee and P. Chandra, Bifurcation analysis and control of Leslie-Gower predator-prey model with Michaelies-Menten type prey harvesting, *Differential Equations and Dynamical Systems,* **20** (2012), 339–366.

[7] R.P. Gupta and P. Chandra, Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelies-Menten type prey harvesting, *Journal of Mathematical Analysis and Applications,* **398** (1)(2013), 278–295.

[8] S. B. Hsu and T. W. Huang, Global stability for a class of predator-prey systems, *SIAM Journal on Applied Mathematics,* **55** (1995), 763–783.

[9] P. Lenzini and J. Rebaza, Nonconstant predator harvesting on ratio-dependent predator-prey models, *Applied Mathematical Sciences,* **4** (16)(2010), 791–803.

[10] J. Lotka, Elements of Physical Biology, *Williams and Wilkins company, Baltimore* (1925).

[11] Y. Lv, Y. Pei, S. Gao and C. Li, Harvesting of a phytoplankton-zooplankton model, *Nonlinear Analysis Real World Applications,* **11** (5)(2010), 3608–3619, .

[12] S. Pal, P. K. Tiwari, A. K. Misra and H. Wang, Fear effect in a three-species food chain model with generalist predator, *Mathematical Biosciences and Engineering* **21** (1)(2023) .

[13] M.K. Singh, B.S. Bhadauria and B.K. Singh, Qualitative analysis of a Leslie-Gower predator-prey system with nonlinear harvesting in predator, *International Journal of Engineering Mathematics* (2016), 1–15 .

[14] D. Xiao and L.S. Jennings, Bifurcation of a ratio-dependent ratio-dependent predator-prey system with constant rate harvesting, *SIAM Journal on Applied Mathematics,***65** (3)(2005), 737–753.

[15] D. Xiao, W. Li and M. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, *Journal of Mathematical Analysis and Applications,* **324** (2006), 14–29.

[16] V. Volterra, Variations and fluctuations of the number of individuals in animal species living together, *Animal Ecology,* (1926), 409–448.

[17] C.R. Zhu and K.Q. Lan, Phase portraits, Hopf bifurcations and limit cycles of Leslie-Gower predator-prey systems with harvesting rates, *Discrete and Continuous Dynamical Systems – B,* **14** (1)(2010), 289-306.