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FUZZY DIRECTED AND WEAKLY DIRECTED BAIRE SPACES

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Abstract. We study fuzzy directed Baire spaces and weakly directed Baire spaces on fuzzy topological spaces. A characterization of fuzzy directed Baire spaces is given using fuzzy G_{δ}- sets. Further, we prove that the weakly directed Baire spaces with a directed Baire spaces. Also we discuss several characterizations of other fuzzy spaces namely fuzzy P- space, fuzzy hyperconnected space.

Keywords: Fuzzy G_{δ}- set, fuzzy nowhere dense set,fuzzy directed Baire space, fuzzy weakly directed Baire space, fuzzy P- space, fuzzy hyperconnected space.

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1. **Introduction**

The concepts of Fuzzy sets were introduced by L.A.Zadeh in 1965 [18]. Then the fuzzy set theory is extension by many researchers. The important concept of fuzzy topological space was offered by C. L. Chang [6] and from that point forward different ideas in topology have been reached out to fuzzy topological space. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

In 1899, Rene Louis Baire [2] introduced the concepts of first and second category sets in his doctoral thesis. In classical topology, Baire space, named in honor of Rene Louis Baire, was first introduced in Bourbaki's [4] Topological generale Chapter IX. The concepts of Baire spaces have been studied extensively in fuzzy topology in [10].

In this paper, we introduce the concept of fuzzy directly Baire space and fuzzy weakly directed Baire spaces. Also, we discuss several characterizations and inter-relations between fuzzy P- space and fuzzy hyperconnected space. Several examples are given to these concepts introduced in this paper.

2. **Preliminaries**

In order to make the exposition self-contained, we give some basic notions and results used in the sequel. In this work by (*X, T*) or simply by *X*, we will denote a fuzzy topological space due to Chang (1968). Let *X* be a non-empty set and *I* the unit interval [0,1]. A fuzzy set λ in *X* is a mapping from *X* into I [18].

Definition 2.1. [6] Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then

(i) $\lambda \vee \mu : X \to [0, 1]$ is defined as follows : $(\lambda \vee \mu)(x) = \text{Max}\,\{\lambda(x), \mu(x)\}$.

(ii) $\lambda \wedge \mu : X \to [0, 1]$ is defined as follows : $(\lambda \wedge \mu)(x) = \text{Min}\{\lambda(x), \mu(x)\}$.

(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

More generally, for a family $\{\lambda_i : i \in I\}$ of fuzzy sets in (X, T), $\psi = \vee_i \lambda_i$ and $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x) : x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x) : x \in X\}$.

Definition 2.2. [6] Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior int(λ) and the closure cl(λ) are defined respectively as follows:

(i). $\text{int}(\lambda) = \vee\{\mu/\mu \le \lambda, \mu \in T\}$.

(ii). $\text{cl}(\lambda) = \wedge\{\mu/\lambda \le \mu, 1 - \mu \in T\}$.

Lemma 2.3. *[1] For a fuzzy set λ of a fuzzy topological space X,*

(i). $1 - int(\lambda) = cl(1 - \lambda)$,

(ii). $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.4. [12] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda<\mu<1$.

Definition 2.5. [12] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu<cl(\lambda)$. That is., $\text{int}[\text{cl}(\lambda)] = 0$, in (X, T).

Definition 2.6. [3] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ -set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$.

Definition 2.7. [3] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ -set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$.

Definition 2.8. [12] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.9. [15] Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.10. [12] A fuzzy topological space (X, T) is called fuzzy first category if the fuzzy set 1_X is a fuzzy first category set in (X, T). That is, $1_X = \bigvee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Otherwise (X, T) will be called a fuzzy second category space

Lemma 2.11. *[1] For a family $A = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\bigvee(cl(\lambda_\alpha)) \leq cl(\bigvee(\lambda_\alpha))$. In case A is a finite set, $\bigvee(cl(\lambda_\alpha)) = cl(\bigvee(\lambda_\alpha))$. Also $\bigvee(int(\lambda_\alpha)) \leq int(\bigvee(\lambda_\alpha))$.*

Definition 2.12. [14] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ-nowhere dense set, if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.13. [15] Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int[\bigvee_{i=1}^{\infty}(\lambda_i)] = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

Definition 2.14. [5] A fuzzy topological space (X, T) is said to be fuzzy hyper-connected if every non-null fuzzy open subset of (X, T) is fuzzy dense set in (X, T). That is a fuzzy topological space (X, T) is hyper-connected if $cl(\mu_i) = 1$, for all $\mu_i \in T$

Definition 2.15. [17] A fuzzy topological space (X, T) is called a fuzzy P- space if each fuzzy G_δ - set in (X, T) is fuzzy open set in (X, T).

3. **Fuzzy Directed and Weakly Directed Baire Spaces**

Definition 3.1. Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy directed Baire space if $(\bigvee_{i=1}^{\infty} int(\lambda_i)) = 0$, where (λ_i)'s are fuzzy G_δ - sets and fuzzy nowhere dense sets in (X, T).

Example 3.2. Let X={*a, b, c*} and the fuzzy sets $\alpha, \beta, \gamma, \vartheta, \lambda$ and μ defined on X as follows

α: X → [0,1] is defined by α(a)=0.3, α(b)=0.6, α(c)=0.5

β: X → [0,1] is defined by β(a)=0.4, β(b)=0.7, β(c)=0.3

γ: X → [0,1] is defined by γ(a)=0.8, γ(b)=0.5, γ(c)=0.4

ϑ: X → [0,1] is defined by ϑ(a)=0.6, ϑ(b)=0.7, ϑ(c)=0.8

λ: X → [0,1] is defined by λ(a)=0.7, λ(b)=0.6, λ(c)=0.7

μ: X → [0,1] is defined by μ(a)=0.8, μ(b)=0.6, μ(c)=0.7

Then

T={$0, \alpha, \beta, \gamma, \lambda \vee \beta, \beta \vee \gamma, \alpha \vee \gamma, \alpha \wedge \beta, \beta \wedge \gamma, \alpha \wedge \gamma, \alpha \wedge \beta \wedge \gamma, 1$} and

T={$0, \lambda, \mu, \vartheta \vee \lambda, \vartheta \vee \mu, \vartheta \wedge \lambda, 1$} is a fuzzy topology on X. Then $intcl(1 - \mu) = int(1 - \mu) = 0$, $intcl(1 - \vartheta \vee \lambda) = int(1 - \vartheta \vee \lambda) = 0$. Therefore the fuzzy nowhere dense sets are $1-\mu$, $1-(\vartheta \vee \lambda)$. There fuzzy sets are $(\bigvee_{i=1}^{\infty}(\alpha \wedge (\alpha \wedge \beta) \wedge (\alpha \wedge \gamma)) = \alpha \wedge \beta \wedge \gamma$ is a fuzzy G_δ -set in (X,T). This implies that $int[(\bigvee_{i=1}^{\infty}(\alpha \wedge \beta \wedge \gamma) \vee (1 - \mu) \vee (1 - \vartheta \vee \lambda)] = 0$. Hence (X,T) is a fuzzy directed Baire Space.

Definition 3.3. Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy weakly directed Baire space if $(\bigvee_{i=1}^{\infty} (\lambda_i)) \neq 0$, where (λ_i)'s are fuzzy G_δ - sets and fuzzy nowhere dense sets in (X, T).

Example 3.4. Let X={a, b, c} and the fuzzy sets $\lambda, \mu, \gamma, \alpha, \beta$ and ϑ defined on X as follows

λ: X → [0,1] is defined by λ(a)=0.6, λ(b)=0.7, λ(c)=0.5

μ: X → [0,1] is defined by μ(a)=0.6, μ(b)=0.6, μ(c)=0.8

γ: X → [0,1] is defined by γ(a)=0.7, γ(b)=0.7, γ(c)=0.6

α: X → [0,1] is defined by α(a)=0.6, α(b)=0.7, α(c)=0.6

β: X → [0,1] is defined by β(a)=0.6, β(b)=0.5, β(c)=0.6

ϑ: X → [0,1] is defined by ϑ(a)=0.7, ϑ(b)=0.6, ϑ(c)=0.7

Then T={$0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \wedge \mu, 1$} and T= {$0, \alpha, \beta, \vartheta, \alpha \vee \beta, \beta \vee \vartheta, \alpha \vee \vartheta, \alpha \wedge \beta, \beta \wedge \vartheta, 1$}.

Then $intcl(1 - \alpha) = int(1 - \alpha) = 0$

$intcl(1 - \beta) = int(1 - \beta) = 0$. The fuzzy sets are $\bigvee_{i=1}^{\infty}[(\lambda \wedge \mu) \wedge (\mu \wedge \gamma)] = \lambda \wedge \mu$ is a fuzzy G_δ-sets in (X, T).

This implies that int $(\bigvee_{i=1}^{\infty}(\lambda \wedge \mu) \wedge (1 - \alpha) \vee (1 - \beta)) = \alpha \wedge \beta \neq 0$. Hence (X,T) is a fuzzy directed Baire Space.

Proposition 3.5. *Let (X,T) be a fuzzy topological space. Then the following are equivalent.*

(i) (X,T) is a fuzzy directed Baire space.

(ii) int(λ) = 0, for every fuzzy first category set λ in (X,T).

(iii) cl(μ) = 1,for each fuzzy residual set μ in (X,T).

Proof. (i) ⇒ (ii)

Let λ be a fuzzy first category set in (X,T). Then $\lambda_i = (\bigvee_{i=1}^{\infty}(\lambda_i))$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Then $int(\lambda) = int(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$. Since (X,T) is a fuzzy directed Baire space, $int(\bigvee_{i=1}^{\infty} (\lambda_i))$ =0. Hence int(λ) = 0 for fuzzy first category set λ in (X,T).

(ii) ⇒ (iii)

Let μ be a fuzzy residual set. Then (1 - μ) is a fuzzy first category set in (X,T). By hypothesis, int(1-μ) = 0. Then by Lemma (2.3) 1 - cl(μ) = 0. Hence cl(μ) = 1, for fuzzy residual set μ in (X,T).

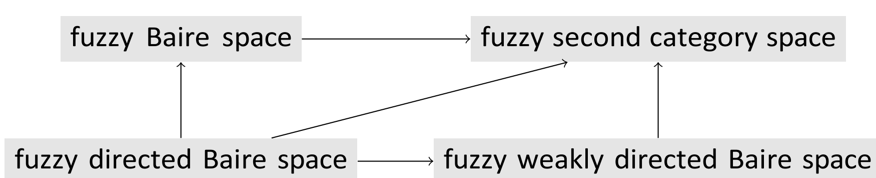
(iii) $\Longrightarrow$ (i)
Let λ be a fuzzy first category set in (X,T). Then $\lambda = (\bigvee_{i=1}^{\infty}(\lambda_i))$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Now λ is a fuzzy first category set implies that $(1 - \lambda)$ is a fuzzy residual set in (X,T). By hypothesis, we have $cl(1 - \lambda) = 1$ then by Lemma(2.3), $1 - int(\lambda) = 1$. Hence $int(\lambda) = 0$. That is, $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Hence (X,T) is a fuzzy directed Baire space. □

Proposition 3.6. *If the fuzzy topological space (X,T) is a fuzzy directed Baire space. Then (X,T) is a fuzzy second category space.*

Proof. Let (X,T) be a fuzzy directed Baire space. Then $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are G_δ - sets and fuzzy nowhere dense set in (X,T). Then $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$ (otherwise $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$ implies that $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = int(1) = 1$ and hence $0 = 1$, a contradiction). Hence (X,T) is fuzzy second category space. □

Proposition 3.7. *If the fuzzy P - space (X,T) is a fuzzy hyperconnected space, then (X,T) is a fuzzy directed Baire space.*

Proof. Let λ be a fuzzy G_δ - set in a fuzzy P- space (X,T). Since (X,T) is a fuzzy P- space is a fuzzy open set in (X,T). Since the fuzzy P- space (X,T) is a fuzzy hyper connected space, then fuzzy open set λ in (X,T) is a fuzzy dense set in (X,T).That is $cl(\lambda) = 1$. Hence λ is a fuzzy G_δ- set and fuzzy nowhere dense set in (X,T). Then by the proposition[3.5], $(1 - \lambda)$ is a fuzzy first category set in (X,T). Therefore $(1 - \lambda) = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Then, $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = int(1 - \lambda) = 1 - cl(\lambda) = 1 - 1 = 0$. Hence we have $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Also, λ is a fuzzy G_δ - sets in (X,T). Therefore (X,T) is a fuzzy directed Baire space.
The following arrow diagram shows the relation between the four spaces namely, fuzzy Baire space, fuzzy second category, fuzzy directed Baire space, fuzzy weakly directed Baire space. □



Proposition 3.8. *If the fuzzy first category set λ in a fuzzy directed Baire space (X,T) is a fuzzy closed set, Then $int(cl(\lambda)) = 0$ in (X,T).*

Proof. Let λ be a fuzzy first category in a fuzzy topological space (X,T).
Since (X,T) fuzzy directed Baire space, by Proposition 3.5, int(λ)=0. Now λ is a fuzzy closed set int (X,T), implies that cl(λ)=λ. Hence int cl(λ) = int(λ) = 0. Therefore int(cl(λ) = 0) in (X,T). □

Proposition 3.9. *If λ is any fuzzy first category set and G_δ - set in a fuzzy Baire space (X,T). Such that cl(λ) = λ, then (X,T) is a fuzzy directed Baire Space.*

Proof: The proof follows from the above.

Proposition 3.10. *If λ be a fuzzy set in a fuzzy directed Baire space (X,T) then 1-λ is fuzzy first category set in (X,T).*

Proof. Let λ be a fuzzy set in a fuzzy directed Baire space (X,T). Since λ is a intersection of fuzzy G_δ - sets in (X,T), $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i) \in$ T and since λ is a fuzzy dense set in (X,T), cl(λ) = 1. Then cl($\bigvee_{i=1}^{\infty}(\lambda_i)$) = 1. But cl($\bigvee_{i=1}^{\infty}(\lambda_i)$) $\leq$ ($\bigvee_{i=1}^{\infty} cl(\lambda_i)$).
Hence $1 \leq \bigvee_{i=1}^{\infty} cl(\lambda_i)$. That is, $\bigvee_{i=1}^{\infty} cl(\lambda_i) = 1$. Then we have cl($\lambda_i$) = 1, for each $\lambda_i \in T$ and hence cl(int(λ_i))=1, which implies that 1 - cl(int(λ_i))=0 and hence intcl($1 - \lambda_i$))=0. Therefore 1-λ_i is a fuzzy nowhere dense set in (X,T). Now 1-λ = 1- $\bigvee_{i=1}^{\infty}(\lambda_i) = \bigwedge_{i=1}^{\infty}(1-\lambda_i)$. Therefore 1-$\lambda$ = $\bigwedge_{i=1}^{\infty}(1-\lambda_i)$ where (1-λ_i)'s are fuzzy nowhere dense sets in (X,T). Hence 1-λ is a first category set in (X,T). □

Proposition 3.11. *If λ be a fuzzy set in a fuzzy directed Baire space (X,T) then λ is fuzzy residual set in (X,T).*

Proof. Let λ be a fuzzy set in a fuzzy directed Baire space (X,T), since intersection of fuzzy $G\ \delta$ - sets is fuzzy dense set in (X,T). By Prop 3.5, we have 1-λ is a fuzzy first category set in (X,T) and hence λ is fuzzy residual set in (X,T). □

Proposition 3.12. *If the fuzzy topological space (X,T) is a fuzzy Directed Baire space, then (X,T) is a fuzzy Baire space.*

Proof. Let λ be a fuzzy set in a fuzzy directed Baire space (X,T), then λ is a intersection of fuzzy G_δ- sets is fuzzy dense set in (X,T). By Prop [3.5], 1-λ is a fuzzy first category set in (X,T) and $1-\lambda$ = ($\bigwedge_{i=1}^{\infty}(1-\lambda_i)$), where ($1-\lambda_i$)'s are fuzzy nowhere dense set in (X,T). But int($1-\lambda$) = 1-cl(λ) =1-1=0. (Since λ is fuzzy dense),cl(λ)=1. Then int($\bigwedge_{i=1}^{\infty}(1-\lambda_i)$) = $int(1-\lambda) = 0$ and hence (X,T) is a fuzzy Baire space. □

Proposition 3.13. *If the fuzzy topological space (X,T) is a fuzzy Directed Baire space, then* $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$*, where* (λ_i)*'s are fuzzy dense and fuzzy open sets in (X,T).*

Proof. Let λ be a fuzzy set in a fuzzy directed Baire space (X,T). Then $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy open sets in (X,T). Now, $cl(\lambda) = cl[\bigvee_{i=1}^{\infty}(\lambda_i)]$ in (X,T). But $cl[\bigvee_{i=1}^{\infty}(\lambda_i)] \leq [\bigvee_{i=1}^{\infty} cl(\lambda_i)]$, implies that $1 \leq [\bigvee_{i=1}^{\infty} cl(\lambda_i)]$. That is, $\bigvee_{i=1}^{\infty} cl(\lambda_i) = 1$ and then $cl(\lambda_i) = 1$ in (X,T). Hence $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy dense and fuzzy open sets in (X,T). □

Proposition 3.14. *If* λ *is a fuzzy set in a fuzzy Directed Baire space (X, T) such that* $\mu \leq 1 - \lambda$*, where* μ *is a fuzzy* F_σ *- set in (X, T), then* μ *is a fuzzy* σ *- nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy set in a fuzzy directed Baire space. Since intersection fuzzy G_δ- sets in (X,T), is fuzzy dense set, This implies that λ is a fuzzy dense set in (X, T) such that $\mu \leq 1 - \lambda$. Now $\mu \leq (1 - \lambda) \Longrightarrow int(\mu) \leq int(1 - \lambda) = 1 - cl(\lambda) = 1 - 1 = 0$ and hence $int(\mu) = 0$. Therefore μ is a fuzzy σ - nowhere set in (X, T). □

Proposition 3.15. *If* λ *is a fuzzy set in a fuzzy Directed Baire space (X, T) such that* $\mu \leq 1 - \lambda$*, where* μ *is a fuzzy* F_σ *- set in (X,T), then* μ *is a fuzzy* σ *- nowhere dense set in (X, T). (X, T) is a fuzzy* σ *- Baire space.*

Proof. The above Prof 3.14, Let μ is a fuzzy σ nowhere dense set in (X, T). Hence (X,T) is a fuzzy σ - Baire space. □

Theorem 3.16. *[10] In a fuzzy topological space (X,T), a fuzzy set* λ *is nowhere dense set if and only if* $1 - \lambda$ *is a fuzzy dense and fuzzy* G_δ*- set in (X, T).*

Proposition 3.17. *If a fuzzy topological space (X,T) is a fuzzy Volterra space, then* $int(\bigwedge_{i=1}^{N}(\lambda_i)) = 0$*, where* (λ_i)*'s are fuzzy nowhere dense sets in (X,T).*

Proof. Let (λ_i)'s [i=1 to N] be fuzzy nowhere dense sets in (X,T). Then by Theorem 3.16 $(1-\lambda_i)$'s are fuzzy dense and G_δ- sets in (X,T).
Since (X,T) is a fuzzy Volterra space, $cl(\bigvee_{i=1}^{N}(1 - \lambda_i)) = 1$. Now $cl(\bigvee_{i=1}^{N}(1 - \lambda_i)) = 1$ implies that $cl(1 - \bigwedge_{i=1}^{N}(\lambda_i)) = 1$. Then we have $1 - int(\bigwedge_{i=1}^{N}(1 - \lambda_i)) = 1$. This implies $int(\bigwedge_{i=1}^{N}(\lambda_i)) = 0$. □

Proposition 3.18. *If a fuzzy topological space (X,T) is a fuzzy Directed Baire space, then (X,T) is a fuzzy Volterra space.*

Proof. Let (X,T) be a fuzzy directed Baire Space. then $int(\bigvee_{i=1}^{N}(\lambda_i)) = 0$ where (λ_i)'s are fuzzy G_δ- sets and fuzzy nowhere dense sets in (X,T). Now $int(\bigvee_{i=1}^{N}(\lambda_i)) \leq int(\bigvee_{i=1}^{\infty}(\lambda_i))$ implies that $int(\bigvee_{i=1}^{N}(\lambda_i)) = 0$. Then $1 - int(\bigvee_{i=1}^{N}(\lambda_i)) = 1 - 0 = 1$. Then we have $cl(1 - \bigvee_{i=1}^{N}(\lambda_i)) = 1$. This implies that $cl(\bigvee_{i=1}^{N}(\lambda_i)) = 1$. Since (λ_i)'s [i=1 to N] are fuzzy G_δ - sets and fuzzy nowhere dense sets in (X,T) by Theorem 3.16 $(1-\lambda_i)$'s are fuzzy dense and fuzzy G_δ - sets in (X,T). Hence $cl(\bigvee_{i=1}^{N}(1 - \lambda_i)) =$, where $(1-\lambda_i)$'s are fuzzy dense and fuzzy G_δ - sets in (X,T) implies that (X,T) is a fuzzy Volterra space. □

Remark 3.19. A fuzzy Baire space need not be a fuzzy Directed Baire space. For consider, the following example:

λ: X → [0,1] is defined by λ(a)=0.8, λ(b)=0.6, λ(c)=0.7

μ: X → [0,1] is defined by μ(a)=0.6, μ(b)=0.9, μ(c)=0.8

γ: X → [0,1] is defined by γ(a)=0.7, γ(b)=0.5, γ(c)=0.9

α: X → [0,1] is defined by α(a)=0.6, α(b)=0.5, α(c)=0.5

β: X → [0,1] is defined by β(a)=0.5, β(b)=0.4, β(c)=0.4

ϑ: X → [0,1] is defined by ϑ(a)=0.4, ϑ(b)=0.5, ϑ(c)=0.4

Then T={0, λ, μ, γ, $(\lambda \vee \mu)$, $(\lambda \wedge \mu)$, $(\lambda \wedge \gamma)$, $(\lambda \vee \gamma)$, $(\mu \wedge \gamma)$, $\lambda \wedge (\mu \vee \gamma)$, $\lambda \vee (\mu \wedge \gamma)$, $\mu \wedge (\lambda \vee \gamma)$, $\mu \vee (\lambda \wedge \vee)$, $\gamma \wedge (\lambda \vee \mu)$, $\gamma(\lambda \wedge \mu)$, $(\lambda \wedge \mu \wedge \gamma)$, $(\lambda \vee \mu \vee \gamma)$, 1} is a fuzzy topology on X. Then T={0, α, β, δ, $(\alpha \vee \beta)$, $(\beta \vee \vartheta)$, $(\alpha \wedge \beta)$, $(\beta \wedge \vartheta)$, $\vartheta \wedge (\alpha \vee \beta)$, 1} Now, $(1-\lambda)$, $(1-\mu)$, $(1-\gamma)$,$(1-\lambda \vee \mu)$ are fuzzy nowhere dense sets and $\alpha \vee \beta$, $\beta \vee \vartheta$ are fuzzy G_δ- sets in (X,T).
Then $\delta = \{(1 - \lambda) \vee (1 - \mu) \vee (1 - \gamma) \vee (\alpha \vee \beta)\} \neq 0$, and $\eta = int[(1 - \lambda) \vee (1 - \mu) \vee (1 - \gamma) \vee (1 - \lambda \vee \mu)] = 0$. Hence (X,T) is not a fuzzy directed Baire space whereas (X,T) is a fuzzy Baire space.

Proposition 3.20. *If the fuzzy topological space (X,T) is a fuzzy first category space, Then (X,T) is not a fuzzy directed Baire space.*

Proof. Let (X, T) is a fuzzy first category space. Then $(\bigvee_{i=1}^{\infty}(\lambda_i)) = 1_x$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Now for the fuzzy first category set 1_x, we have $intcl(1_x) \neq 0$. Hence (X,T) is not a fuzzy directed Baire space. □

Theorem 3.21. *[9] If λ be a fuzzy dense and fuzzy G_δ- set in a fuzzy topological space (X,T), then $(1 - \lambda)$ is a fuzzy first category set in (X, T).*

Proposition 3.22. *If a fuzzy topological space(X, T) has a fuzzy dense and fuzzy G_δ- set, then (X, T) is not a fuzzy directed Baire space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ- set in (X, T). Then by Theorem 3.21 , 1 - λ is a first category set in (X, T). Now $intcl(1 - \lambda) = 1 - clint(\lambda) > 1 - cl(\lambda) > 1 - 1 = 0$ (Since $cl(\lambda) = 1$). Hence we have intcl$(1 - \lambda) \neq 0$, for a fuzzy first category set $(1 - \lambda)$ in (X,T). Therefore (X,T) is not a fuzzy directed Baire space. □

Proposition 3.23. *If every fuzzy residual set in a fuzzy P-space (X,T) contains a non-zero fuzzy dense and fuzzy G_δ - set in (X,T), then (X,T) is a fuzzy Directed Baire space.*

Proof. Let λ be a fuzzy first category in (X,T). Then $(1 - \lambda)$ is a fuzzy residual set in (X,T). Then there exists a fuzzy G_δ - set μ in (X,T) such that $\mu \leq (1 - \lambda)$. Then $\lambda \leq (1 - \mu)$. Now $intcl(\lambda) \leq intcl(1 - \mu)$, implies that $intcl(\lambda) \leq 1 - clint(\mu)$. Since (X,T) is a fuzzy P- space, the fuzzy G_δ- set, μ is fuzzy open in (X,T) and $int(\mu) = \mu$. Therefore we have $intcl(\lambda) \leq 1 - cl(\mu) = 1 - 1 = 0$. (Since μ is a fuzzy dense). Then we have $intcl(\lambda) = 0$ and hence λ is a fuzzy nowhere dense set and fuzzy G_δ- set in (X,T). Hence $int(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$ □

Theorem 3.24. *[16] If(X, T) is a fuzzy open hereditarily irresolvable space, any fuzzy σ-nowhere dense set in (X, T) is a fuzzy nowhere dense set in (X, T).*

Proposition 3.25. *If the fuzzy topological space (X, T) is a fuzzy directed Baire space and fuzzy open hereditarily irresolvable space (X,T), then (X, T) is a fuzzy Baire Space.*

Proof. Let (X, T)be a fuzzy directed Baire Space and fuzzy open hereditarily irresolvable space. Then, $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy G_δ- sets and fuzzy nowhere dense sets in (X,T). Hence $int(\bigvee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Therefore (X,T) is a fuzzy Baire space. □

4. **Conclusion**

In this study, we explored the structural and topological properties of fuzzy directed Baire spaces, extending classical Baire space theory into the fuzzy and directed setting. By integrating the concepts of fuzzy topology and directed sets, we were able to generalize key properties such as completeness and category in a more flexible, graded context. Our findings demonstrate that fuzzy directed Baire spaces retain many essential characteristics of traditional Baire spaces while offering a richer framework for handling uncertainty and gradation.

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