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SOME RESULTS ON HOLOMORPHIC FUNCTIONS BASED ON ERROR FUNCTION

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Abstract : In this paper, we introduce a new subclass of uniformly convex functions with negative coefficients defined by error function. We obtain the coefficient bounds, growth distortion properties, extreme points and radii of close-to-convexity and starlikeness for functions belonging to the class $TS(\upsilon, \varrho)$. Furthermore, we obtained modified Hardamard product, convolution and integral operators for this class.

Keywords: analytic; coefficient bound; starlike; convolution; extreme points.

2020 Subject classification: 30C45, 30C50.

Let A denote the class of all functions u of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

in the open unit disc $E = \{z \in \mathbb{C}: |z| < 1\}$. Let S be a subclass of A with univalent and normalized by $u(0) = u'(0) - 1 = 0$. A function $u \in A$ is starlike function of the order ξ $(0 \leq \xi < 1)$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \xi, \quad z \in E \tag{1.2}$$

and convex function of the order ξ $(0 \leq \xi < 1)$, if it satisfies

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \xi, \quad z \in E. \tag{1.3}$$

Also, the classes of starlike and convex functions are denoted by $S^*(\xi)$ and $K(\xi)$ respectively. For $u \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), \ \ (z \in E). \tag{1.4}$$

Note that $u * g \in A$.

Let T denotes the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \ \ (z \in E) \tag{1.5}$$

and let $T^*(\xi) = T \cap S^*(\xi),\ C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [16].

One of the most widely applied functions studied in recent years is the error function which is used in partial differential equations physics as well as in probability science, statistics and applied mathematics. Properties and a series of inequalities related to this function can be seen in [3]. The error function can appears in most cases due to the normal curve. The inverse of this function was also introduced by Carlitz [4]. Also, because researchers have been able to define the Taylor series of this function as the series form of a normalized analytic function, they paved the way for this function based on analytic univalent functions. The motivation for introducing a certain subclass was based on the error function and its properties.

$$\begin{aligned} E_r u(z) &= \frac{2}{\pi}\int_0^z e^{-t^2}\, dt \\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)n!} \\ &= \frac{2}{\sqrt{\pi}}\left(-\frac{z^3}{3}+\frac{z^5}{10}-\frac{z^7}{42}+\cdots\right), \end{aligned} \tag{1.6}$$

was introduced in [1] and was studied in [3, 5] and [7](See also [2]). After modification of (1.1), Ramachandran et al [13]. introduced the error function as

$$E_r u(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}z^n \qquad (z \in C\colon\ |z| < 1) \tag{1.7}$$

Now, we consider the functions

$$\begin{aligned} M(z) &= [(2z - E_r u) * (2z - E_r u)] \\ &= z - \sum_{n=2}^{\infty}\frac{1}{[(2n-1)(n-1)!]^2}z^n \end{aligned} \tag{1.8}$$

and

$$L_g u(z) = (M * u)(z) = z - \sum_{n=2}^{\infty} \phi\,(n) a_n z^n \qquad (1.9)$$

where $\phi(n) = \frac{1}{[(2n-1)(n-1)!]^2}$.

We now, define a new subclass of functions belonging to the class A.

Definition 3. For $-1 \le \upsilon < 1$ and $\varrho \ge 0$, we let $TS(\upsilon, \varrho)$ be the subclass of A consisting of functions of the form (1.5) and satisfying the analytic criterion

$$\Re\left\{\frac{z\left(L_g u(z)\right)'}{L_g u(z)} - \upsilon\right\} \ge \varrho \left|\frac{z\left(L_g u(z)\right)'}{L_g u(z)} - 1\right|, \qquad (1.10)$$

for $z \in E$.

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, distortion properties, extreme points, radii of close-to-convex, starlike, Hardmard product, convolution and integral operators for the class.

2. Coefficient bounds

In this section, we obtain a necessary and sufficient condition for function $u(z)$ is in the class (υ, ϱ) .

Theorem 2.1. The function u defined by (1.5) is in the class $TS(\upsilon, \varrho)$ iff

$\sum_{n=2}^{\infty}[n(1+\varrho) - (\upsilon + \varrho)]\,\phi(n)|a_n| \le 1 - \upsilon, \qquad (2.1)$

where $-1 \le \upsilon < 1, \varrho \ge 0$.

The result is sharp.

Proof. We have $u \in TS(\upsilon, \varrho)$ if and only if the condition (2.1) satisfied. Upon the fact that

$$\Re(w) > \varrho|w-1| + \upsilon \Leftrightarrow \Re\{w\left(1+\varrho e^{i\theta}\right) - \varrho e^{i\theta}\} > \upsilon,\ -\pi \leq \theta \leq \pi.$$

Equation (1.10) may be written as

$$\begin{aligned} & \Re\left\{\frac{z\left(L_g u(z)\right)'}{L_g u(z)}\left(1+\varrho e^{i\theta}\right) - \varrho e^{i\theta}\right\} \\ = \quad & \Re\left\{\frac{z\left(L_g u(z)\right)'1+\varrho e^{i\theta}) - \varrho e^{i\theta} L_g u(z)}{L_g u(z)}\right\} > \ \upsilon. \end{aligned} \tag{2.2}$$

Now, we let

$$\begin{aligned} A(z) &= z\left(L_g u(z)\right)'1+\varrho e^{i\theta}) - \varrho e^{i\theta} L_g u(z) \\ B(z) &= L_g u(z). \end{aligned}$$

Then,

$$|A(z) + (1-\upsilon)B(z)| > |A(z) - (1+\upsilon)B(z)|,\ \text{for } 0 \leq \upsilon < 1.$$

For $A(z)$ and $B(z)$ as above, we have

$$|A(z)+(1-\upsilon)B(z)| \geq (2-\upsilon)|z| - \sum_{n=2}^{\infty}[n+1-\upsilon+\varrho(n-1)]\,\phi(n)|a_n||z^n|$$

and similarly

$$|A(z)-(1+\upsilon)B(z)| \leq \upsilon|z| - \sum_{n=2}^{\infty}[n-1-\upsilon+\varrho(n-1)]\,\phi(n)|a_n||z^n|.$$

Therefore

$$|A(z)+(1-\upsilon)B(z)| - |A(z)-(1+\upsilon)B(z)|$$

$$\geq\; 2(1-\upsilon) - 2\sum_{n=2}^{\infty}[n-\upsilon+\varrho(n-1)]\,\phi(n)|a_n|$$

or $$\sum_{n=2}^{\infty}[n-\upsilon+\varrho(n-1)]\,\phi(n)|a_n| \leq (1-\upsilon),$$

which yields (2.1).

On the other hand, we must have

$$\Re\left\{\frac{z\left(L_g u(z)\right)'}{L_g u(z)}\left(1+\varrho e^{i\theta}\right) - \varrho e^{i\theta}\right\} \geq \upsilon.$$

Upon choosing the values of z on the positive real axis where $0 < |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{n=2}^{\infty}\left[n-\upsilon+\varrho e^{i\theta}(n-1)\right]\phi(n)|a_n|\, r^n}{z - \sum_{n=2}^{\infty}\phi(n)\,|a_n|\, r^n}\right\} \geq 0.$$

Since $\Re\left(-e^{i\theta}\right) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{n=2}^{\infty}[n-\upsilon+\varrho(n-1)]\,\phi(n)|a_n|\, r^n}{z - \sum_{n=2}^{\infty}\phi(n)\,|a_n|\, r^n}\right\} \geq 0.$$

Letting $r \to 1^-$, we get the desired result. Finally the result is sharp with the extremal function u given by

$$u(z) = z - \frac{1-\upsilon}{[n(1+\varrho)-(\upsilon+\varrho)]\phi(n)} z^n \qquad (2.3)$$

3 Growth and Distortion Theorems

Theorem 3.1. Let the function u defined by (1.5) be in the class $TS(\upsilon,\varrho)$. Then for $|z| = r$, we have

$$r - \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(2)} r^2 \leq |u(z)| \leq r + \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(2)} r^2. \qquad (3.1)$$

Equality holds for the function $u(z) = z - \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(2)} z^2$ (3.2)

Proof. We only prove the right hand side inequality in (3.1), since the other inequality can be justified using similar arguments. In view of Theorem 2.1, we have

$\sum_{n=2}^{\infty} |a_n| \leq \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(2)}$.

Since ,

$$\begin{aligned} u(z) &= z - \sum_{n=2}^{\infty} a_n z^n \\ |u(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leq r + \sum_{n=2}^{\infty} |a_n| r^n \\ &\leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq r + \sum_{n=2}^{\infty} \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(2) r^2} \end{aligned}$$

which yields the right hand side inequality of (3.1).

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. Let the function u defined by (1.5) be in the class $TS(\upsilon,\varrho)$. Then for $|z| = r$

$$1 - \frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(2)r} \le |u'(z)| \le 1 + \frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(2)r}.$$

Equality holds for the function given by (3.2).

Theorem 3.3. If $u \in TS(\upsilon,\varrho)$ then $u \in TS(\gamma)$, where

$$\gamma = 1 - \frac{(n-1)(1-\upsilon)}{[n-\upsilon+\varrho(n-1)]\phi(2)-(1-\upsilon)}$$

Equality holds for the function given by (3.2).

Proof. It is sufficient to show that (2.1) implies

$$\sum_{n=2}^{\infty}(n-\gamma)\,|a_n| \le 1-\gamma,$$

that is

$$\frac{n-\gamma}{1-\gamma} \le \frac{[n-\upsilon+\varrho(n-1)]\phi(n)}{(1-\upsilon)},$$

then

$$\gamma \le 1 - \frac{(n-1)(1-\upsilon)}{[n-\upsilon+\varrho(n-1)]\phi(n)-(1-\upsilon)}.$$

The above inequality holds true for $n \in \mathbb{N}_0, n \ge 2, \varrho \ge 0$ and $0 \le \upsilon < 1$.

4 Extreme points

Theorem 4.1. Let $u_1(z) = z$ and

$$u_n(z) = z - \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n, \tag{4.1}$$

for $n = 2,3,\cdots$. Then $u(z) \in TS(\upsilon,\varrho)$ if and only if $u(z)$ can be expressed in the form $u(z) = \sum_{n=1}^{\infty} \zeta_n u_n(z)$, where $\zeta_n \geq 0$ and $\sum_{n=1}^{\infty} \zeta_n = 1$.

Proof. Suppose $u(z)$ can be expressed as in (4.1). Then

$$\begin{aligned} u(z) &= \sum_{n=1}^{\infty} \zeta_n u_n(z) = \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n u_n(z) \\ &= \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n \left\{ z - \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n \right\} \\ &= \zeta_1 z + \sum_{n=2}^{\infty} \zeta_n z - \sum_{n=2}^{\infty} \zeta_n \left\{ \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n \right\} \\ &= z - \sum_{n=2}^{\infty} \zeta_n \left\{ \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{n=2}^{\infty} \zeta_n \left(\frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} \right) \left(\frac{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon} \right) \\ &= \sum_{n=2}^{\infty} \zeta_n = \sum_{n=1}^{\infty} \zeta_n - \zeta_1 = 1 - \zeta_1 \leq 1. \end{aligned}$$

So, by Theorem 2.1, $u \in TS(\upsilon,\varrho)$.

Conversely, we suppose $u \in TS(\upsilon,\varrho)$. Since

$$|a_n| \leq \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}, \quad n \geq 2.$$

We may set

$$\zeta_n = \frac{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}|a_n|, \quad n \geq 2$$

and $\zeta_1 = 1 - \sum_{n=2}^{\infty} \zeta_n$. Then

$$\begin{aligned} u(z) &= z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \zeta_n \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n \\ &= z - \sum_{n=2}^{\infty} \zeta_n [z - u_n(z)] = z - \sum_{n=2}^{\infty} \zeta_n z + \sum_{n=2}^{\infty} \zeta_n u_n(z) \\ &= \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n u_n(z) = \sum_{n=1}^{\infty} \zeta_n u_n(z). \end{aligned}$$

Corollary 4.2. The extreme points of $TS(\upsilon, \varrho)$ are the functions

$u_1(z) = z$ and $u_n(z) = z - \frac{1-\upsilon}{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)} z^n,\ n \geq 2.$

5 Radii of Close-to-convexity, Starlikeness and Convexity

Theorem 5.1. Let $u \in TS(\upsilon, \varrho)$. Then u is close-to-convex of order δ *in* $|z| < R_1$, where

$R_1 = \inf_{n\geq 2} \left[\frac{(1-\delta)[n-\upsilon+\varrho(n-1)]\phi(n)}{n(1-\upsilon)}\right]^{\frac{1}{n-1}}.$

The result is sharp with the extremal function u is given by (2.3).

Proof. It is sufficient to show that $|u'(z) - 1| \leq 1 - \delta$, for $|z| < R_1$. We have

$$|u'(z) - 1| = \left| -\sum_{n=2}^{\infty} n\, a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}.$$

Thus $|u'(z) - 1| \leq 1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} |a_n||z|^{n-1} \leq 1. \tag{5.1}$$

But Theorem 2.1, confirms that

$$\sum_{n=2}^{\infty} \frac{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon} |a_n| \leq 1. \tag{5.2}$$

Hence (5.1) will be true if

$$\frac{n|z|^{n-1}}{1-\delta} \leq \frac{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}.$$

We obtain

$$|z| \leq \left[\frac{(1-\delta)[n-\upsilon+\varrho(n-1)]\phi(n)}{n(1-\upsilon)}\right]^{\frac{1}{n-1}}, n \geq 2$$

as required.

Theorem 5.2. *Let $u \in TS(\upsilon,\varrho)$. Then u is starlike of order δ in $|z| < R_2$, where*

$$R_2 = \inf_{n\geq 2} \left[\frac{(1-\delta)[n-\upsilon+\varrho(n-1)]\phi(n)}{(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

Proof. We must show that $\left|\frac{zu\prime(z)}{u(z)} - 1\right| \leq 1-\delta$, for $|z| < R_2$. We have

$$\begin{aligned}\left|\frac{zu'(z)}{u(z)}-1\right| &= \left|\frac{-\sum_{n=2}^{\infty}(n-1)\,a_n z^{n-1}}{1-\sum_{n=2}^{\infty}a_n\,z^{n-1}}\right| \\ &\le \frac{\sum_{n=2}^{\infty}(n-1)\,|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n|\,|z|^{n-1}} \\ &\le \quad 1-\delta. \end{aligned} \tag{5.3}$$

Hence (5.2) holds true if

$$\sum_{n=2}^{\infty}(n-1)\,|a_n||z|^{n-1} \le (1-\delta)\left(1-\sum_{n=2}^{\infty}|a_n|\,|z|^{n-1}\right)$$

or equivalently,

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|a_n||z|^{n-1} \le 1. \tag{5.4}$$

Hence, by using (5.2) and (5.4) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{[n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}$$

$$\Rightarrow |z| \le \left[\frac{(1-\delta)[n-\upsilon+\varrho(n-1)]\phi(n)}{(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}, n \ge 2$$

which completes the proof.

By using the same technique in the proof of Theorem 5.2. Then we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let* $u \in TS(\upsilon,\varrho)$. *Then* u *is convex of order* δ *in* $|z| < R_3$, *where*

$R_3 = \inf_{k\ge 2}\left[\frac{(1-\delta)[n-\upsilon+\varrho(n-1)]\phi(n)}{n(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}.$

The result is sharp with the extremal function u is given by (2.3).

6 Inclusion theorem involving modified Hadamard products. For functions

$$u_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}| z^n, \; j = 1,2 \tag{6.1}$$

in the class A, we define the modified Hadamard product $u_1 * u_2(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$u_1 * u_2(z) = z - \sum_{n=2}^{\infty} |a_{n,1}| \, |a_{n,2}| z^n.$$

We can prove the following.

Theorem 6.1. *Let the function* $u_j,\ j = 1,2$, *given by* (6.1) *be in the class* $TS(\upsilon, \varrho)$ *respectively. Then* $u_1 * u_2(z) \in TS(\upsilon, \varrho, \xi)$, *where*

$$\xi \leq 1 - \frac{(n-1)(1+\varrho)(1-\upsilon)^2}{[n-\upsilon+\varrho(n-1)]^2 \phi(n) - (1-\upsilon)^2}, \; n \geq 2.$$

Proof. Employing the technique used earlier by Schild and Silverman [15], we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{[n-\xi+\varrho(n-1)]\phi(n)}{1-\xi} |a_{n,1}||a_{n,2}| \leq 1.$$

Since $u_j \in TS(\upsilon, \varrho),\ j = 1,2$ then we have

$$\sum_{n=2}^{\infty} \frac{[n-\upsilon+\varrho(n-1)]\phi(n)}{1-\upsilon}|a_{n,1}| \leq 1$$

and $$\sum_{n=2}^{\infty} \frac{[n-\upsilon+\varrho(n-1)]\phi(n)}{1-\upsilon}|a_{n,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[n-\upsilon+\varrho(n-1)]\phi(n)}{1-\upsilon}\sqrt{|a_{n,1}||a_{n,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[n-\xi+\varrho(n-1)]\phi(n)}{1-\xi}|a_{n,1}||a_{n,2}| \leq \frac{[n-\upsilon+\varrho(n-1)]\phi(n)}{1-\upsilon}\sqrt{|a_{n,1}||a_{n,2}|}, \; n \geq 2,$$

that is

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{(1-\xi)[n-\upsilon+\varrho(n-1)]}{1-\upsilon)[n-\xi+\varrho(n-1)]}.$$

Note that

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{(1-\upsilon)}{[n-\upsilon+\varrho(n-1)]\phi(n)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[n-\upsilon+\varrho(n-1)]\phi(n)} \leq \frac{(1-\xi)[n-\upsilon+\varrho(n-1)]}{1-\upsilon)[n-\xi+\varrho(n-1)]}, \; n \geq 2,$$

or, equivalently, that

$$\xi \leq 1 - \frac{(n-1)(1+\varrho)(1-\upsilon)^2}{[n-\upsilon+\varrho(n-1)]^2\phi(n)-(1-\upsilon)^2}, \; n \geq 2.$$

Since

$$A(k) = 1 - \frac{(n-1)(1+\varrho)(1-\upsilon)^2}{[n-\upsilon+\varrho(n-1)]^2\phi(n)-(1-\upsilon)^2}, \; n \geq 2$$

is an increasing function of $n, n \geq 2$, letting $n = 2$ in last equation, we obtain

$$\xi \leq A(2) = 1 - \frac{(1+\varrho)(1-\upsilon)^2}{[2-\upsilon+\varrho]^2\phi(2) - (1-\upsilon)^2}.$$

Finally, by taking the function given by (2.3), we can see that the result is sharp.

7 Convolution and Integral Operators

Let $u(z)$ be defined by (1.5) and suppose that

$g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$. Then,the Hadamard product of $u(z)$ and $g(z)$ defined here by

$$u(z) * g(z) = u * g(z) = z - \sum_{n=2}^{\infty} |a_n|\,|b_n| z^n.$$

Theorem 7.1. Let $u \in TS(\upsilon, \varrho)$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n, 0 \leq |b_n| \leq 1$. Then $u * g \in TS(\upsilon, \varrho)$.

Proof. In view of Theorem 2.1, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} [n - \upsilon + \varrho(n-1)]\, \Phi_n(\mu, b, \alpha) |a_n||b_n| \\ \leq \quad & \sum_{n=2}^{\infty} [n - \upsilon + \varrho(n-1)]\, \Phi_n(\mu, b, \alpha) |a_n| \\ \leq \quad & (1-\upsilon). \end{aligned}$$

Theorem 7.2. *Let* $u \in TS(\upsilon, \varrho, \mu, b, \alpha)$ and α be real number such that $\alpha > -1$. Then the function $F(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} u(t)dt$ also belongs to the class $TS(\upsilon, \varrho)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| \, z^n, \text{ where } A_n = \left(\frac{\alpha+1}{\alpha+n}\right) |a_n|.$$

Since $\alpha > -1$, than $0 \leq A_n \leq |a_n|$. Which in view of Theorem 2.1, $F \in TS(\upsilon, \varrho)$.

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