Some Arrival Cases in Quorum Queues under N-Policy

Haneen Hayder Mahmoud Hussein Kashifalghitaa[1], Ali Hussein Mahmood Al-Obaidi[2*]
[1]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq, haneen.hayder.pure322@student.uobabylon.edu.iq
[2]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq, aalobaidi@uobabylon.edu.iq

Abstract

This paper examines quorum queueing systems with N-policy by considering different arrivals cases. In this model, the server is idle when the queue size is less than r; otherwise, it operates R units or less. The probability generating function of queue size derives precisely under various distributions of arrivals. The random walk is analyzed using the first excess level theory.

Keywords: quorum queue, N-policy, first excess level theory, service batches, bulk input, marked delayed renewal process, delayed renewal process, point process, marked point process, random walk.

1. Introduction

Lots of judgments must be made within the environment of randomness. Random failures of equipment, fluctuating industrial rates, and new requirements are all components of everyday decision-making processes [1]. Several requests in the industry, such as systems of digital communication, systems of computer networks, and systems of inventory control, involve the server being idle when the system has things less than level r and returning when the stuffs in it achieve a certain tolerance. Perhaps, in a production system, the manufacturer does not start while waiting for specific basic r of units collected in the system throughout the idle period [2].

The innovative study of bulk arrival queueing system with N-policy was created by Lee and Srinivasan [3]. Besides other studies, they accomplished a scheme to get the optimal stationary functioning strategy under an appropriate linear cost structure. Next, Lee et al. [4] deemed this kind broadly during numerous methods. In regard, some ideas of this structure were examined by Lee et al. [5], Teghem [6], Medhi [7], Kalita and Choudhury [8], Choudhury and Baruah [9], Ali and Al-Obaidi [10], and Kazem and Al-Obaidi [11].

Bailey [12] initially deliberated queueing systems with batch service. This model was developed by Chaudhry and Templeton, where the server suspensions until the number of arrival units achieves a fixed level r, and they gave the name quorum for this model [13]. Many investigations are performed under bulk queueing systems. For example, Dshalalow and Tadj [14] analyzed the queues with fixed accumulation levels. Moreover, Abolnikov and Dshalalow [15] examined the service with delayed queueing systems. Furthermore, Dshalalow [16] studied queueing systems under q-policy.

This article presents the results for the probability generating function (in short, pgf) of the number of units under the N-policy queue. The system is the analysis of random walk by the first excess level theory. Three cases discuss corresponding to different distributions of arrivals to find the pgf of queue size.

1.1. Preliminaries

Throughout this work, we give the following essential definitions and notations. In addition, there are some theorems, and results are displayed without proof in this content. The tools in this section are subject to various efforts related to Dshalalow [17, 18, 19, 20, 21], and all are seen in the same given at work by Ali [10] and Kazem [11].

Definition 1.1.1 [10, 11]

A collection $\{F_t : t \geq 0\}$ of sub σ-algebra of $F(\Omega)$ is called a filtration if this collection is nondecreasing monotone.

Definition 1.1.2 [10, 11]

The stochastic process $\{X_t\}$ is called F_t-adapted process if for every Borel set $A \subseteq R_0^+$ such that $\{\nu : X(t,\nu) \in A\} \in F_t$.

Definition 1.1.3 [10, 11]

A probability space $(\Omega, F(\Omega), P)$ with filtration $\{F_t\}$ is said to be filtration probability space and denoted by $(\Omega, F(\Omega), P, F_t)$.

Definition 1.1.4 [10, 11]

A random variable T defining on filtration probability space $(\Omega, F(\Omega), P, F_t)$ is said to be a stopping time if $\{t \leq T\} \in F_t$.

Definition 1.1.5 [10, 11]

A point process (arrival time) is a.s. monotone increasing sequence stopping time $\{T_n : n = 1,2,3, \dots\}$ on R_0^+.

Definition 1.1.6 [10, 11]

The sum of all stopping time points in the interval $[0, t]$ is termed as counting point process and is given in the following formula

$$C_t = C([0,t]) = \sum_{n=1}^{\infty} 1_{[0,t]}(T_n) = \sum_{n=1}^{\infty} \epsilon_{T_n}([0,t])$$

where $1_{[0,t]}(T_n)$ and $\epsilon_{T_n}([0,t])$ are called the indicator function and unit mass (Dirac) measure, respectively defined on the interval $[0,t]$ as the following

$$1_{[0,t]}(T_n) = \epsilon_{T_n}([0,t]) = \begin{cases} 1, & T_n \in [0,t] \\ 0, & T_n \notin [0,t] \end{cases}$$

Definition 1.1.7 [17, 20]

A counting point process N_t has stationary increments property if its increments $N_{t+s} - N_t$ for every $t > 0$ has the same distribution.

Definition 1.1.8 [18, 21]

A counting point process N_t has independent increments property if its increments $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_1} - N_{N_{t_0}}, \dots, N_{t_{n+1}} - N_{t_n}$ are independent for $0 \le t_0 < t_1 < \cdots < t_{n+1} < \infty$.

Definition 1.1.9 [10, 11]

The point process $T = \{t_i : i = 1,2,\dots\}$ and associated counting point process N_t is called the Poisson point process and Poisson counting process, respectively, if the counting process N_t has Poisson distribution with parameter (rate or intensity of process) λt and its increments $N_{t+1} - N_t$ for $t > 0$ has stationary and independent distributed.

Definition 1.1.10 [10, 11]

Let $T = \{t_i : i = 1,2,\dots\}$ be a point process with a counting point process N_t and let $X = \{X_n : i = 1,2,\dots\}$ be a sequence of iid real-valued random variables with probability generating function (in short, pgf) $a(z)$. Then $(X,T) = \{(X_n, T_n) : n = 1,2,\dots\}$ is called the marked point process, and if X is independent of T, then (X,T) is called the marked point process with position independent marking. However, if the mark X_n may depend on the inter-arrival times $\Delta_n = T_n - T_{n-1}$, then (X,T) is called the marked point process with position dependent marking. (note: X_n on Δ_n is conditionally independent of X_i for $i < n$).

Definition 1.1.11 [10, 11]

The counting point process in the interval $[0,t]$ associated with the marked point process given in the above definition is called the counting marked point process, and it is given as the following

$$M_t = M([0,t]) = \sum_{n=1}^{\infty} X_n \epsilon_{T_n}([0,t])$$

Definition 1.1.12 [10, 11]

The marked process $(X, T) = \{(X_n, T_n): n = 1,2, \ldots\}$ and its counting process M_t are called the marked Poisson process and marked Poisson counting process with independent marking, respectively, if the counting process has independent and stationary increments and its pgf is a compound Poisson distribution.

Definition 1.1.13 [10, 11]

(i) Let T be a nonnegative random variable; then the Laplace-Stieltjes transform is given as

$$\beta(\theta) = E\left[e^{-\theta T}\right].$$

(ii) The moment generating function $\mathcal{M}$ is given as

$$\mathcal{M}(\theta) = E\left[e^{\theta T}\right] = \beta(-\theta)$$

Proposition 1.1.14

If the random variable T is independent of the Poisson counting process C_T, then

$$E\left[z^{C_T} e^{-\theta T}\right] = \beta\big(\theta + \lambda(1 - z)\big).$$

Proof:

The pgf of Poisson r.v. N_t with parameter λ is given as

$$E[z^{N_t}] = e^{\lambda t(z-1)}$$

Then we have

$$E\left[z^{C_T} e^{-\theta T}\right] = E\left[E\left[z^{C_T} e^{-\theta T} | T\right]\right] = E\left[e^{-\theta T} E\left[z^{C_T} | T\right]\right] = E\left[e^{-\theta T} e^{\lambda T(z-1)}\right] = E\left[e^{-(\theta + \lambda(1-z))T}\right]$$
$$= \beta\big(\theta + \lambda(1 - z)\big) \blacksquare$$

Proposition 1.1.15

If the random variable T is independent of the marked Poisson counting process M_T, then

$$E\left[z^{M_T} e^{-\theta T}\right] = \beta\left(\theta + \lambda\big(1 - a(z)\big)\right).$$

Proof:

The pgf of compound Poisson r.v. M_t with parameter λ is given as

$$E[z^{M_t}] = e^{\lambda t(a(z)-1)}$$

Then we have

$$E\left[z^{M_T}e^{-\theta T}\right] = E\left[E\left[z^{M_T}e^{-\theta T}|T\right]\right] = E\left[e^{-\theta T}E[z^{M_T}|T]\right] = E\left[e^{-\theta T}e^{\lambda T(a(z)-1)}\right]$$
$$= E\left[e^{-(\theta+\lambda(1-a(z)))T}\right] = \beta\left(\theta + \lambda\left(1-a(z)\right)\right) \blacksquare$$

1.2. The bulk input of queues [18, 20, 21, 22]

The marked point process characterizes the bulk input of the queue. Consider the marked Poisson point process with position independent marking $(X,T) = \{(X_n, T_n): n = 1,2, \dots\}$ with Poisson point process $T = \{T_n: n = 1,2, \dots\}$ and the arrival rate is λ. Obviously, the random size of units in batches is subjected to the marks X_n for $n = 1,2, \dots$ and these marks are iid nonnegative integer-valued r.v.'s with the pgf $a(z)$ and the expected value a. Moreover, the marks X_n and its position T_n are independent for every n. Consequently, the random size of customers in the line until the nth customer leaving at a time T_n is given as

$$Q_{n+1} = \begin{cases} Q_n + A_{n+1} - 1, & Q_n > 0 \\ X_{B_n} + A_{n+1} - 1, & Q_n = 0 \end{cases}$$

such that X_{B_n} is the first bulk entering the line after the time T_n. Accordingly to Abolnikov and Dukhovny [22], the time-homogeneous Markov chain $\{Q_n\}$ is embedded in $\{Q(t)\}$ and Δ_2-matrix. That is, the process $\{Q_n\}$ is **irreducible**, **aperiodic**, and **recurrent positive** (**ergodic**) because of $P_1'(1-) < 1$. To see that, we have

$$P_i(z) = E[z^{Q_1}|Q_0 = i] = \begin{cases} E\left[z^{i+A_1-1}\right], & i > 0 \\ E\left[z^{X_{B_0}+A_1-1}\right], & i = 0 \end{cases} = \begin{cases} z^{i-1}E[z^{A_1-1}], & i > 0 \\ z^{-1}E[z^{X_{B_0}}]E[z^{A_1}], & i = 0 \end{cases}$$
$$= \begin{cases} z^{i-1}\beta\left(\lambda\left(1-a(z)\right)\right), & i > 0 \\ z^{-1}a(z)\beta\left(\lambda\left(1-a(z)\right)\right), & i = 0 \end{cases}$$

and

$$P_1'(1-) = \beta'(0)\,(-\lambda)a'(1) = ab\lambda = \rho < 1, \;\; a = a'(1) = EX_1, b = ET_1 = -\beta'(0)$$

where ρ is called the **offered load**. Therefore, the pgf of the stationary distribution for the chain $\{Q_n\}$ is given as the following

$$P(z) = \sum_{i=0}^{\infty} p_i\, P_i(z) = p_0 z^{-1}a(z)\beta\left(\lambda\left(1-a(z)\right)\right) + z^{-1}\beta\left(\lambda\left(1-a(z)\right)\right)\sum_{i=1}^{\infty} p_i\, z^i$$

and the Pollaczed-Khinchine formula of this pgf is

$$P(z) = p_0\beta\left(\lambda\big(1-a(z)\big)\right)\frac{a(z)-1}{z-\beta\left(\lambda\big(1-a(z)\big)\right)}$$

where $p_0 = 1 - \rho$.

1.3. Random walk analysis [17, 18, 19, 20, 21]

Let M_t be a renewal delay marked counting process corresponding to the renewal delay marked point process with position dependent marking $(X,T) = \{(X_n, T_n): n = 0, 1, 2, \dots\}$. The delayed renewal process refers to the nature of the distribution of T is not specific, and the inter-renewal times $\{\Delta_n = T_n - T_{n-1}: n = 0, 1, 2, \dots\}$ are independent and identically distributed except $T_0 = \triangle_0$ has a different distribution. Suppose that $M = \{M_n: M_n = M_{t_n} = \sum_{i=0}^{n} X_i\}$ is a random walk.

To avoid clustering in the point process $T = \{T_n: n = 0,1,2, \dots\}$, we assume that T is a nondecreasing monotone sequence. Consequently, the counting process corresponding with this point process will be continuous in probability.

Since $X = \{X_n: n = 0,1,2, \dots\}$ be a sequence of nonnegative random variables. Then we define the following joint transformations as below

$$\beta_0(z,\theta) = Ez^{X_0}e^{-\Delta_0\theta}, |z| \le 1, Re(\theta) \ge 0$$

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\Delta_1\theta}, |z| \le 1, Re(\theta) \ge 0$$

We are interested in examining the performance of the random walk M when the marked counting process M_k achieves fixed stage L. Thus, we will define the following random variables:

(i) The index of first passage time is

$$\alpha = \inf\{k: M_k \ge L\} = \inf\{k: X_0 + \cdots + X_k \ge L\}$$ (***Error! No text of specified style in document.***.1)

(ii) The first passage time (exit time) is T_α and
(iii)The first excess level is M_α.

To do this investigation, we should derive the following joint transform

$$\omega_\alpha = \omega(z_1, z_2, z_3, \theta, \vartheta) = Ez_1^{\alpha} z_2^{M_{\alpha-1}} z_3^{M_\alpha} e^{-\vartheta T_{\alpha-1} - \theta T_\alpha}$$ (***Error! No***

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corresponding to the auxiliary family of random indices

$$\{\alpha(n) \coloneqq \inf\{k: X_0 + \cdots + X_k \geq n\}, n = 0,1,2,\ldots\}$$

(***Error! No text of specified style in document.***.3)

and the family of functionals

$$\{\omega_{\alpha(n)} = E z_1^{\alpha(n)} z_2^{M_{\alpha(n)-1}} z_3^{M_{\alpha(n)}} e^{-\vartheta T_{\alpha(n)-1} - \theta T_{\alpha(n)}}, n = 0,1,2,\ldots\}$$

(***Error! No text of specified style in document.***.4)

by using the D_n operator defined as

$$D_n\{g(n)\}(y) \coloneqq \sum_{n=0}^{\infty} y^n (1-y) g(n), \|y\| < 1$$

(***Error! No text of specified style in document.***.5)

and its inverse D^n defined as

$$D_y^n\big(D_s\{g(s)\}(y)\big) = f(n), n = 0,1,2,\ldots$$

(***Error! No text of specified style in document.***.6)

where

$$s \longmapsto D_y^s(\varphi(y,z)) = \begin{cases} \lim\limits_{y\to 0} \dfrac{1}{s!} \dfrac{\partial^s}{\partial y^s} \left[\dfrac{1}{1-y} \varphi(y,z)\right], s \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad , s < 0 \end{cases}$$

(***Error! No text of specified style in document.***.7)

It is clear that $\alpha = \alpha(L-1)$ from equations (1.3.1-1.3.4), then applying the operator D_n on the functional $\omega_{\alpha(n)}$ and the inverse of this operator D^{L-1} on $D_n(\omega_{\alpha(n)})$ can restore $\omega_{\alpha(L-1)}$ to ω_α.

The next theorem will consider the essential characteristics of the inverse operator D^n, which serve us in progress sections.

Theorem 1.3.1 [18, 20, 21]

Let D^n be an inverse operator defined in the equation (1.3.7). Then the following features are accurate:

(i) D^n is a linear functional.

(ii) $D_y^n\big(1(y)\big) = 1,$ where $1(y) = 1$ for all $y \in \mathbb{R}$

(iii) Let h be an analytic function at zero. Then, it holds true that

$$D_y^n\left(y^i h(y)\right) = D_y^{n-i} h(y).$$

(***Error! No text of specified style in document.***.8)

(iv) In particular of (iii), if $i = n$, we have

$$D_y^n\big(y^n h(y)\big) = h(0).$$

(***Error! No text of specified style in document.***.9)

(v) Let $h(y) = \sum_{j=0}^{\infty} h_j y^j$. Then,

$$D_y^n\big(h(y)\big) = \sum_{j=0}^{n} h_j \;\; and \;\; D_y^n\big(h(ay)\big) = \sum_{j=0}^{n} h_j a^j$$

(***Error! No text of specified style in document.***.10)

(vi) For any real number s, it holds true that

$$D_y^n\left\{\frac{1}{1-sy}\right\} = \begin{cases} \dfrac{1-s^{n+1}}{1-s}, & s \neq 1 \\ n+1, & s = 1 \end{cases}$$

(***Error! No text of specified***

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(vii) For any real number s and for a positive integer m, except for $s = m = 1$, it holds true that

$$D_y^n\left\{\frac{1}{(1-sy)^m}\right\} = \begin{cases} \sum_{i=0}^{n} \binom{m+i-1}{i} s^i & \text{except for } s = m = 1 \\ n+1, & s = m = 1 \end{cases}$$

(***Error! No text of specified style in document.***.12)

$(viii)$ For two real numbers s and r and a positive integer m, it holds true that

$$D_y^n\left\{\frac{1}{1-ry}\frac{1}{(1-sy)^m}\right\} = \begin{cases} \frac{1}{1-r}\sum_{i=0}^{n} \binom{m+i-1}{i} \left(s^i - r^{n+1}\left(\frac{s}{r}\right)^i\right), r \neq 1 \\ \sum_{i=0}^{n} \binom{m+i-1}{i} s^j (n-i+1), & r = 1 \end{cases}$$

(***Error! No text of specified style in document.***.13)

The subsequent theorem is fundamental to our further issues.

Theorem 1.3.2 (The Key of Random Walk Analysis) (First Excess Level Theory) [17, 18, 19]

Let the next functionals be given as

$$\eta_0 \coloneqq \beta_0(xz_2z_3, \vartheta + \theta), \qquad \eta \coloneqq \beta_1(xz_2z_3, \vartheta + \theta),$$

$$\chi_0 \coloneqq \beta_0(xz_3, \theta), \qquad \chi \coloneqq \beta_1(xz_3, \theta),$$

$$\chi_0^1 \coloneqq \beta_0(z_3, \theta), \qquad \chi^1 \coloneqq \beta_1(z_3, \theta).$$

Then, it holds true that

$$\omega^*(x) = D_n(\omega_{\alpha(n)}(x)) = \chi_0^1 - \chi_0 + \frac{\eta_0 z_1}{1 - \eta z_1}(\chi^1 - \chi)$$

(***Error! No text of specified style in document.***.14)

and the functional ω_α is given as the following

$$\omega_\alpha = \omega_\alpha(z_1, z_2, z_3, \vartheta, \theta) = E z_1^\alpha z_2^{M_{\alpha-1}} z_3^{M_\alpha} e^{-\vartheta T_{\alpha-1} - \theta T_\alpha},$$

$$= D_x^{L-1}\left(\chi_0^1 - \chi_0 + \frac{\eta_0 z_1}{1 - \eta z_1}(\chi^1 - \chi)\right)$$

(***Error! No text of specified style in document.***.15)

Corollary 1.3.3 [17, 18, 19]

Let $z_1 = z_2 = 1, \vartheta = 0$, then

$$\varpi_\alpha = \varpi_\alpha(z_3, \theta) = \omega_\alpha(1,1, z_3, 0, \theta) = E z_3^{M_\alpha} e^{-\theta T_\alpha}$$

$$= \beta_0(z_3, \theta) - \left(1 - \beta_1(z_3, \theta)\right) D_x^{L-1}\left(\frac{\beta_0(x z_3, \theta)}{1 - \beta_1(x z_3, \theta)}\right)$$

(***Error! No text of specified style in document.***.16)

and if $T_0 = \Delta_0 = 0$ and $X_0 = M_0 = i \geq 0$, then $\beta_0(z_3, \theta) = z_3^i$ and

$$\varpi_\alpha = \varpi_\alpha(z_3, \theta) = E z_3^{M_\alpha} e^{-\theta T_\alpha}$$

$$= z_3^i - z_3^i\left(1 - \beta_1(z_3, \theta)\right) D_x^{L-1}\left(x^i \frac{1}{1 - \beta_1(x z_3, \theta)}\right)$$

(***Error! No text of specified style in document.***.17)

1.4. N-Policy on queues [17, 18, 20, 21]

This system has the rule to serve the customers. The policy is that the host stops working if no persons are in line. However, this server stays idle if the line does not contain a fixed number L of customers. Therefore, the busy time starts when the number of customers becomes L or more in the queue. This system with the N-policy is subjected to the random walk analysis when $i = 0$ and by eq. (19), we have

$$\varpi_\alpha = \varpi_\alpha(z, \theta) = \omega_\alpha(1,1, z, 0, \theta) = E z^{M_\alpha} e^{-\theta T_\alpha} = 1 - \left(1 - \beta_1(z, \theta)\right) D_x^{L-1}\left(\frac{1}{1 - \beta_1(x z, \theta)}\right)$$

Suppose that the system is modelling such that the r.v. X_i is independent of exponential r.v. Δ_i with parameter λ for all $i = 1,2, \ldots$ In other words, the marked point process with position independent marking, so we have

$$\beta_1(z, \theta) = E z^{X_1} e^{-\Delta_1 \theta} = E z^{X_1} E e^{-\Delta_1 \theta} = a(z)\beta(\theta) = a(z)\frac{\lambda}{\lambda + \theta}$$

and the pgf of M_α is

$$\psi(z) \coloneqq \omega_\alpha(1,1,z,0,0) = Ez^{M_\alpha} = 1 - [1 - a(z)]\, D_x^{L-1}\left(\frac{1}{1 - a(xz)}\right)$$

To derive Kendall's formula for this system, we begin with

$$Q_{n+1} = \begin{cases} M_\alpha + A_{n+1} - 1, & Q_n = 0 \\ Q_n + A_{n+1} - 1, & Q_n > 0 \end{cases}$$

and then we get

$$P_i(z) = E[z^{Q_1} | Q_0 = i] = \begin{cases} z^{-1}\psi(z)\beta\big(\lambda - \lambda a(z)\big), & i = 0 \\ z^{i-1}\beta\big(\lambda - \lambda a(z)\big), & i > 0 \end{cases}$$

Thus, the pgf of this probability distribution of this model

$$P(z) = \sum_{i=0}^{\infty} p_i P_i(z) = p_0 \psi(z) z^{-1} \beta\big(\lambda - \lambda a(z)\big) + z^{-1}\beta\big(\lambda - \lambda a(z)\big)\sum_{i=1}^{\infty} p_i z^i$$

By doing some steps, we have

$$P(z) = p_0 \beta\big(\lambda - \lambda a(z)\big)\frac{\psi(z) - 1}{z - \beta\big(\lambda - \lambda a(z)\big)}$$

and

$$p_0 = \frac{1 - \lambda ab}{\psi} = \frac{1 - \rho}{\psi}$$

where

$$\psi{:} = EM_\alpha = \psi'(z)|_{z=1}$$

2. A quorum queue.

This system with a quorum policy happens when the single server is subject to batch service. That means the server is busy with R units at the same time. Moreover, the input of the queue is bulk arrivals. We are interested in discussing this queue with N-policy and generalizing it to multiple vacations.

2.1. A quorum queue with N-policy.

Assume the server is working under the quorum condition and the input queue is bulk. Moreover, the server becomes idle when the number of units in the line is less than $r \geq 1$. Furthermore, the capacity of service is $R \geq r$, and the server resume when the buffer contains r or more units.

Let the input batch M_t be modelled as a marked Poisson with position independent marking with joint transform

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\theta T_1} = Ez^{X_1}Ee^{-\theta T_1} = a(z)\frac{\lambda}{\lambda+\theta}$$

Suppose that the server rests up when the queue size $Q_n = i < r$ at T_n and resumes when the line size of arrivals reaches level r or higher. Let $X_0 = i$ at $T_0 = 0$, then

$$\beta_0(z,\theta) = Ez^{X_0}e^{-\theta T_0} = Ez^{i}e^{-\theta(0)} = z^i$$

The length of the idle period in $[T_n, T_n + T_\alpha]$ is the first passage of time T_α and the functional $\psi(z)$ is given as the following

$$\psi(z) \coloneqq Ez^{M_\alpha} = z^i - z^i[1-a(z)]\, D_x^{L-1-i}\left(\frac{1}{1-a(zx)}\right), r > i$$

To find Kendall's formula for this model, we start with

$$Q_{n+1} = \begin{cases}(M_\alpha - R)^+ + A_{n+1}, & Q_n < r\\ (Q_n - R)^+ + A_{n+1}, & Q_n \geq r\end{cases}$$

and then we have

$$P_i(z) = E[z^{Q_1}|Q_0 = i] = \begin{cases}\beta\big(\lambda - \lambda a(z)\big)Ez^{(M_\alpha - R)^+}, & i < r\\ \beta\big(\lambda - \lambda a(z)\big)Ez^{(i-R)^+}, & i \geq r\end{cases}$$

$$= \begin{cases}\beta\big(\lambda - \lambda a(z)\big)Ez^{(M_\alpha - R)^+}, & i < r\\ \beta\big(\lambda - \lambda a(z)\big)z^{(i-R)^+}, & i \geq r\end{cases}$$

To appearance a work [23], we see

$$Ez^{(M_\alpha - R)^+} = D_x^R(Ex^{M_\alpha} + z^{-R}[Ez^{M_\alpha} - E(zx)^{M_\alpha}])$$

$$= D_x^{R-i}[1-[1-a(x)]\, D_y^{r-1-i}\left(\frac{1}{1-a(xy)}\right)] + z^{i-R}(1-[1-a(z)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zy)}\right))$$
$$- z^{i-R}D_x^{R-i}(1-[1-a(zx)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zxy)}\right))$$

To discuss the case $R = r$, we have the following

$$Ez^{(M_\alpha-R)^+}=Ez^{(M_\alpha-r)^+}=Ez^{M_\alpha-r}=z^{-r}Ez^{M_\alpha}=z^{-r}\left[z^i-z^i[1-a(z)]\,D_x^{L-1-i}\left(\frac{1}{1-a(zx)}\right)\right]$$
$$=z^{i-r}-z^{i-r}[1-a(z)]\,D_x^{L-1-i}\left(\frac{1}{1-a(zx)}\right)$$

Let $\psi_0^i(z)\coloneqq Ez^{(M_\alpha-R)^+}$, then the pgf of Q_n is

$$P(z)=\sum_{i=0}^{\infty}p_iP_i(z)=\sum_{i=0}^{r-1}p_i\beta\big(\lambda-\lambda a(z)\big)Ez^{(M_\alpha-R)^+}+\sum_{i=r}^{\infty}p_i\beta\big(\lambda-\lambda a(z)\big)z^{(i-R)^+}$$
$$=\beta\big(\lambda-\lambda a(z)\big)\sum_{i=0}^{r-1}p_i\psi_0^i(z)+\beta\big(\lambda-\lambda a(z)\big)\sum_{i=r}^{\infty}p_iz^{(i-R)^+}$$

Since $(i-R)^+=0$ when $i\le R$ and $r<R$, then

$$P(z)=\beta\big(\lambda-\lambda a(z)\big)\sum_{i=0}^{r-1}p_i\psi_0^i(z)+\beta\big(\lambda-\lambda a(z)\big)\sum_{i=r}^{R-1}p_i+\beta\big(\lambda-\lambda a(z)\big)\sum_{i=R}^{\infty}p_iz^{i-R}$$
$$=\beta\big(\lambda-\lambda a(z)\big)\left[\sum_{i=0}^{r-1}p_i\psi_0^i(z)+\sum_{i=r}^{R-1}p_i+z^{-R}\sum_{i=R}^{\infty}p_iz^i\right]$$
$$=\beta\big(\lambda-\lambda a(z)\big)\left[\sum_{i=0}^{r-1}p_i\psi_0^i(z)+\sum_{i=r}^{R-1}p_i+z^{-R}\left(P(z)-\sum_{i=0}^{R-1}p_iz^i\right)\right]$$

Therefore, the pgf of Q_n is

$$P(z)=\frac{\beta\big(\lambda-\lambda a(z)\big)}{\left[z^R-\beta\big(\lambda-\lambda a(z)\big)\right]}\left[\sum_{i=0}^{r-1}p_i[z^R\psi_0^i(z)-z^i]+\sum_{i=r}^{R-1}p_i[z^R-z^i]\right]$$

When $r=R$, we have the following

$$P(z)=\frac{\beta\big(\lambda-\lambda a(z)\big)}{\left[z^r-\beta\big(\lambda-\lambda a(z)\big)\right]}\left[\sum_{i=0}^{r-1}p_i[z^r\psi_0^i(z)-z^i]\right]$$

2.2. Applications on a quorum queue with N-policy.

In this section, we will be concerned about the essential objectives of our effort. We plan to obtain pgf of the number of units for this scheme when arriving batches have several formulas as the following cases:

Case I:-

When $a(z) = z$, it is not bulk input because the input is subjected to the ordinary Poisson process. Let $r = 2, R = 3.$ Now, we begin to derive

$$\psi_0^i(z) = D_x^{R-i}\left(1 - [1-x]\, D_y^{r-1-i}\left(\frac{1}{1-xy}\right)\right) + z^{i-R}\left(1 - [1-z]\, D_y^{r-1-i}\left(\frac{1}{1-zy}\right)\right)$$
$$- z^{i-R} D_x^{R-i}\left(1 - [1-zx]\, D_y^{r-1-i}\left(\frac{1}{1-zxy}\right)\right)$$

$$= D_x^{R-i}\left(1 - (1-x)\,\frac{1-x^{r-i}}{(1-x)}\right) + z^{i-R}\left(1 - (1-z)\,\frac{1-z^{r-i}}{(1-z)}\right)$$
$$- z^{i-R} D_x^{R-i}\left(1 - (1-zx)\,\frac{1-(zx)^{r-i}}{(1-zx)}\right)$$

$$= D_x^{R-i}\left(1 - 1 + x^{r-i}\right) + z^{i-R}\left(1 - 1 + z^{r-i}\right) - z^{i-R} D_x^{R-i}\left(1 - 1 + (zx)^{r-i}\right)$$

$$= 1 - D_x^{R-i}\left(x^{r-i}\right) + z^{i-R} z^{r-i} - z^{i-R} D_x^{R-i}\left((zx)^{r-i}\right)$$

$$= 1 - 1 + z^{r-R} - z^{i-R} z^{r-i} D_x^{R-i}\left(x^{r-i}\right)$$

$$= z^{r-R} - z^{r-R}$$

$$= 0$$

Since $r = 2, R = 3$, then we have for all i

$$\psi_0^i(z) = 0$$

and

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{2-1} p_i[z^3\psi_0^i(z) - z^i] + \sum_{i=2}^{3-1} p_i[z^3 - z^i]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{1} p_i[z^3(0) - z^i] + \sum_{i=2}^{2} p_i[z^3 - z^i]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{1} p_i[-z^i] + p_2(z^3 - z^2)\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[-z^0 p_0 - z^1 p_1 + p_2(z^3 - z^2)]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[-p_0 - z p_1 + p_2(z^3 - z^2)]$$

$$= \frac{\beta(\lambda - \lambda z)}{[\beta(\lambda - \lambda z) - z^3]}[p_0 + z p_1 + p_2(z^2 - z^3)]$$

<u>Case II</u>:-

When the arriving batch is type 1 geometrically distributed with parameter p, then $a(z) = \frac{pz}{1-qz}$, $q = 1 - p$. Let $r = 2, R = 3$. Now, we start to obtain

$$\psi_0^i(z) = D_x^{R-i}[1 - \left[1 - \frac{px}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{1 - \frac{pxy}{1-qxy}}\right)] + z^{i-R}(1 - \left[1 - \frac{pz}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{1 - \frac{pzy}{1-qzy}}\right)) - z^{i-R} D_x^{R-i}(1 - \left[1 - \frac{pzx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{1 - \frac{pzxy}{1-qzxy}}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-qx-px}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qxy-pxy}{1-qxy}}\right)] + z^{i-R}(1 - \left[\frac{1-qz-pz}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qzy-pzy}{1-qzy}}\right)) - z^{i-R} D_x^{R-i}(1 - \left[\frac{1-qzx-pzx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qzxy-pzxy}{1-qzxy}}\right))$$

$$= D_x^{R-i}[1 - \frac{1-x^{r-i}}{1-qx} + qx\frac{1-x^{r-i-1}}{1-qx}] + z^{i-R}(1 - \frac{1-z^{r-i}}{1-qz} + qz\frac{1-z^{r-i-1}}{1-qz}) - z^{i-R} D_x^{R-i}(1 - \frac{1-(zx)^{r-i}}{1-qzx} + qzx\frac{1-(zx)^{r-i-1}}{1-qzx})$$

$$= (1-q)D_x^{R-r}[\frac{1}{1-qx}] + \frac{(1-q)z^{r-R}}{1-qz} - (1-q)z^{r-R}D_x^{R-r}(\frac{1}{1-qzx})$$

$$= 1-q^{R-r+1} + \frac{(1-q)z^{r-R} - (1-q)z^{r-R} + (1-q)q^{R-r+1}z}{1-qz}$$

$$\psi_0^i(z) = 1-q^{R-r+1} + \frac{pq^{R-r+1}z}{1-qz} = 1-q^{R-r+1}\left(\frac{1-z}{1-qz}\right)$$

Since we know the pgf of Q_n and it is given as

$$P(z) = \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[p_0[z^3\psi_0^0(z)-1] + p_1[z^3\psi_0^1(z)-z] + p_2[z^3-z^2]\right]$$

where

$$\psi_0^0(z) = \psi_0^1(z) = 1-q^{R-r+1}\left(\frac{1-z}{1-qz}\right)$$

Then

$$P(z) = \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[(p_0+p_1)\left(1-q^{R-r+1}\left(\frac{1-z}{1-qz}\right)\right)z^3 - (p_0+p_1 z + p_2 z^2 - p_2 z^3)\right]$$

<u>Case III</u>:-

When the arriving batch is distributed with pgf $a(z) = pz + qz^2, q = 1-p, 0 < p < 1$. Let $r = 2, R = 3$. We will simplify the following

$$1-a(z) = 1-pz-qz^2 = p+q-pz-qz^2 = (1-qz)(1-z)$$

Now, we start to obtain

$$\psi_0^i(z) = D_x^{R-i}[1-[(1-qx)(1-x)]\, D_y^{r-1-i}\left(\frac{1}{(1-qxy)(1-xy)}\right)] + z^{i-R}(1 - [(1-qz)(1-z)]\, D_y^{r-1-i}\left(\frac{1}{(1-qzy)(1-zy)}\right)) - z^{i-R}D_x^{R-i}(1 - [(1-qzx)(1-zx)]\, D_y^{r-1-i}\left(\frac{1}{(1-qzxy)(1-zxy)}\right))$$

To find the following quantity, we have

$$D_y^{r-1-i}\left(\frac{1}{(1-qzy)(1-zy)}\right) = D_y^{r-1-i}\left(\frac{\frac{1}{p}}{1-qzy}+\frac{\frac{-q}{p}}{1-zy}\right) = \frac{1+\frac{q}{p}(1-q^{r-i-1})z^{r-i}}{(1-qz)(1-z)}$$

So, we can return to our task

$$\psi_0^i(z) = D_x^{R-i}\left[1-\big((1-qx)(1-x)\big)\ \frac{1+\frac{q}{p}(1-q^{r-i-1})x^{r-i}}{(1-qx)(1-x)}\right] + z^{i-R}\left[1-(1-qz)(1-z)\ \frac{1+\frac{q}{p}(1-q^{r-i-1})z^{r-i}}{(1-qz)(1-z)}\right] - z^{i-R}D_x^{R-i}\left[1-(1-qzx)(1-zx)\ \frac{1+\frac{q}{p}(1-q^{r-i-1})(zx)^{r-i}}{(1-qzx)(1-zx)}\right]$$

$$= \frac{q}{p}\left(q^{r-i-1}-1\right)$$

So, we have

$$\psi_0^0(z) = \frac{q}{p}(q^{2-0-1}-1) = \frac{q}{p}(q-1) = -q,\ \ \psi_0^1(z) = \frac{q}{p}(q^{2-1-1}-1) = \frac{q}{p}(0-1) = -\frac{q}{p}$$

Since we know the pgf of Q_n and it is given as

$$P(z) = \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[p_0[-qz^3-1]+p_1\left[-\frac{q}{p}z^3-z\right]+p_2[z^3-z^2]\right]$$

$$= \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[(-qp_0-\frac{q}{p}p_1)z^3-(p_0+zp_1+p_2z^2-p_2z^3)\right]$$

3. Conclusion

In this paper, we derive a general formula of pgf for queue size when the quorum queue is subjected to N-policy. Moreover, we obtain the directed expression for this pgf when the input system has various distributions of arrivals. By using the first-level access theory, these investigations build on the random walk analysis. In this case, the number of units in the queue hits level r to affect the server's job. In fact, the server is active when the line has r components or more and serves R units or less. Otherwise, this server will be idle. We assume that the arrival batches are ordinary Poisson processes, type 1 geometrically distributed, and others kinds of arrival batches.

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