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Dirac equation with spinors using Clifford algebra $Cl_{(3,1)}$ approach

Shabnam Jahan Ansari[1*], V. R. Lakshmi Gorty[2]

[1,2] Department of BSH

Mukesh Patel School of Technology Management & Engineering

SVKM's Narsee Monjee Institute of Management Studies (NMIMS)

Mumbai-400056, India.

*Corresponding author(s). E-mail(s): shabnam.ansari@nmims.edu

Contributing authors: shabnam.ansari@nmims.edu; vrlakshmigorty@nmims.edu

Abstract

The Dirac equation and spinors are developed in Clifford algebra $Cl_{(3,1)}$ using the representation of the Gamma matrices. Gamma and Pauli matrices in $Cl_{(3,1)}$ are used to show applications in Clifford algebra using the Clifford-wavelet function. The graphical representation of Gamma matrices is interpreted. Majorana spinors relation is given with charge conjugation in the Clifford algebra sense.

Keywords: Clifford wavelet function, Clifford algebra, Spinors, Pauli matrices, Transformation.

1 Introduction

In 1878, William Kingdon Clifford united the algebra of Hamilton and Grassmann into a single structure and introduced geometric algebra [1]. In [2] authors have discussed Clifford geometric algebra using multi-vectors. Author in [3] developed the geometries of 3D-space and space-time with mathematical structure to formulate physical theories. Clifford wavelet in $Cl_{(3,0)}$ using similitude group $SIM(3)$ is constructed and properties of Clifford wavelet transform using multi-vector function are shown in [4]. In [5] authors have considered Clifford algebra-valued admissible wavelets and proved properties of wavelet transform using admissible wavelet. Clifford-valued function defined on $\mathbb{R}$ is considered in [6] for representing continuous wavelet transform. In [7] author has explain spinors and chiralities in space-time, initial and final spin states provides particle model of the electron. [8] developed mathematical structure of spinors in Clifford algebra and their relation to complex vector spaces also discussed the representation theory of spin groups.

The study in [8] contributed to the Dirac equation and is fundamental to Quantum Mechanics in terms of CA $Cl_{(3,0)}$. It describes the behavior of spin $(-1/2)$ particles like electrons in relativistic quantum field theory. Development of the Mathematical structure of spinors in CA and their transformations were studied earlier in [7]. This gave a sense of motivation to introduce the Dirac equation using CWF, which was developed. Further study explains spinors and chiralities in space-time correlating CA and CWF in $Cl_{(3,1)}$. CWF that have been established is applied to various transformations to see the behavior change, which was lagging to analysis in earlier work in $Cl_{(3,1)}$.

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2 Clifford algebra in $\mathbb{R}^{(3,1)}$

Definition 2.1 *Multiplication Rules for an Orthonormal Basis of* $\mathbb{R}^{(3,1)}$*: The Clifford geometric algebra* $Cl_{(3,1)}$ *is defined for the vector space* $\mathbb{R}^{(3,1)}$ *with a metric of signature* (3,1)*. This algebra is a 16-dimensional algebra over the reals and is generated by a set of basis elements.*

Consider the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^{(3,1)}$ with the following properties:

The basis vectors $\{e_1, e_2, e_3, e_4\}$ satisfy from [1]:

$$e_\mu e_\nu + e_\nu e_\mu = -2\eta_{\mu\nu} \tag{2.1}$$

where $\eta_{\mu\nu}$ is the metric tensor and $\mu, \nu = 1,2,3,4$ $(\mu, \nu$ re indices that run over the dimensions of the space.)

In $\mathbb{R}^{(3,1)}$, the Minkowski metric tensor is given in (2.2)

$$\eta = \mathrm{diag}(1,1,1,-1). \tag{2.2}$$

Definition 2.2 *Multiplication Rules: Considering the definition of Clifford algebra (CA)* $Cl_{(3,1)}$ *as in [9]:*

i) Vector products: $e_\mu e_\nu = \begin{cases} -e_\nu e_\mu \text{ for } \mu \neq \nu \\ \eta_{\mu\mu} \text{ for } \mu = \nu \end{cases}$

(2.3)

ii) Bi-vector products: $e_{\mu\nu} e_{kl} = \delta_{\nu k} e_{\mu l} - \delta_{\mu l} e_{\nu k} + e_{\mu\nu kl}$, where $e_{\mu\nu kl}$ represents the 4-dimensional pseudo-scalar if indices cover all four dimensions.

Using the definition from (2.3) the table is developed. The orthonormal basis are multiplied with each other where the results are displayed below

2.1 Multiplication Table of basis of $Cl_{(3,1)}$

Table : 2 (i)

	1	e_1	e_2	e_3	e_4	e_{12}	e_{13}	e_{14}
1	1	e_1	e_2	e_3	e_4	e_{12}	e_{13}	e_{14}
e_1	e_1	1	e_{12}	e_{13}	$-e_{14}$	e_2	e_3	$-e_4$
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_{24}$	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_{13}$	$-e_{23}$	1	$-e_{34}$	$-e_4$	$-e_1$	e_2
e_4	e_4	e_{14}	e_{24}	e_{34}	-1	$-e_3$	e_2	e_1
e_{12}	e_{12}	$-e_2$	e_1	e_{14}	e_{24}	-1	e_{34}	$-e_{23}$
e_{13}	e_{13}	$-e_3$	$-e_4$	$-e_1$	e_{14}	$-e_{34}$	-1	e_{12}
e_{14}	e_{14}	e_4	$-e_3$	e_2	e_1	e_{23}	$-e_{12}$	-1

Table 2: (ii)

	e_{23}	e_{24}	e_{34}	e_{123}	e_{124}	e_{1234}
1	e_{23}	e_{24}	e_{34}	e_{123}	e_{124}	e_{1234}
e_1	e_{123}	e_{124}	e_{134}	$-e_{23}$	$-e_{24}$	$-e_{234}$
e_{12}	e_{13}	e_{14}	$-e_{1234}$	$-e_3$	$-e_4$	$-e_{134}$
e_{13}	e_{12}	$-e_{1234}$	e_2	$-e_{34}$	$-e_{24}$	$-e_4$
e_{14}	$-e_{1234}$	e_1	e_{12}	$-e_{23}$	e_3	e_2
e_{23}	-1	e_{123}	e_4	$-e_{14}$	$-e_{134}$	$-e_{34}$
e_{24}	e_{13}	e_{123}	e_3	e_{134}	e_{12}	e_{124}
e_{34}	e_{14}	$-e_{1234}$	e_2	e_1	e_{123}	e_{24}
e_{1234}	e_{234}	$-e_{1234}$	e_4	e_3	e_2	e_{1234}

Table 2: (iii)

	e_1	e_2	e_3	e_4	e_{12}	e_{13}	e_{14}
1	e_1	e_2	e_3	e_4	e_{12}	e_{13}	e_{14}
e_{23}	e_{123}	e_3	e_2	e_{234}	e_{13}	e_{12}	$-e_{1234}$
e_{24}	e_{124}	e_4	e_{234}	e_2	e_{14}	$-e_{1234}$	e_1
e_{34}	e_{134}	e_{234}	e_4	e_3	$-e_{1234}$	e_1	e_2
e_{123}	e_{23}	e_{13}	e_{34}	$-e_{1234}$	e_3	e_{24}	e_{14}
e_{124}	e_{24}	e_{34}	$-e_{1234}$	e_3	e_4	e_{123}	e_{123}
e_{134}	e_{34}	$-e_{1234}$	e_4	e_{13}	e_1	e_2	e_{24}
e_{234}	$-e_{1234}$	e_4	e_3	e_2	e_1	e_{13}	e_{12}
e_{1234}	e_{234}	e_{134}	e_{123}	e_4	e_3	e_{14}	e_2

Table 2: (iv)

	e_{23}	e_{24}	e_{34}	e_{123}	e_{124}	e_{134}	e_{234}	e_{1234}
e_{23}	e_{23}	e_{24}	e_{34}	e_{123}	e_{124}	e_{134}	e_{234}	e_{1234}
e_{24}	$-e_{23}$	e_4	e_{14}	$-e_{23}$	e_{24}	e_{34}	$-e_{234}$	e_{1234}
e_{34}	e_3	$-e_{24}$	e_4	e_{123}	$-e_{124}$	e_{134}	e_{234}	$-e_{1234}$
e_{123}	e_{23}	$-e_{14}$	e_4	e_1	e_{24}	e_{34}	$-e_{23}$	$-e_{1234}$
e_{124}	$-e_{34}$	e_3	$-e_2$	$-e_{13}$	e_{124}	$-e_{123}$	e_{134}	$-e_{1234}$
e_{134}	$-e_{23}$	e_4	$-e_{34}$	e_{24}	e_{234}	e_{1234}	$-e_{124}$	e_{12}
e_{234}	$-e_{34}$	$-e_{14}$	e_3	e_{124}	$-e_{123}$	$-e_{124}$	e_{1234}	e_{12}

3 Clifford wavelet in $\boldsymbol{Cl_{(3,1)}}$

Definition 3.1 ***Clifford-wavelet*** *with respect to the Clifford mother-wavelet defined in* $\psi \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ *from [9]*

$$\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{a^2}\psi\left(r_\Theta^{-1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right) = \frac{1}{a^2}\psi\begin{pmatrix} \frac{y_1(x_1-b_1)}{a} \\ \frac{y_2(x_2-b_2)}{a} \\ \frac{(Dy_3-By_4)(Ax_3e_3+B(x_4e_4-b_3e_3))}{ae_3(AD-BC)} \\ \frac{(-Cy_3+Ay_4)(Cx_3e_3+D(x_4e_4-b_4e_4))}{ae_4(AD-BC)} \end{pmatrix} \tag{3.1}$$

where values of A, B, C and D are given below

$$A = e_{11}^2\cos^2\alpha + e_{21}e_{12}\sin^2\alpha$$
$$B = -e_{11}e_{21}\cos\alpha\,\sin\alpha - e_{21}e_{22}\,\sin\alpha\,\cos\alpha$$
$$C = -e_{12}e_{11}\,\sin\alpha\,\cos\alpha - e_{22}e_{12}\sin\alpha\,\cos\alpha$$
$$D = e_{12}e_{21}\sin^2\alpha + e_{22}^2\cos^2\alpha$$

and also note that

$$y_1 = x_1e_1$$
$$y_2 = x_2e_2$$
$$y_3 = Ax_3e_3 + Bx_4e_4$$
$$y_4 = Cx_3e_3 + Dx_4e_4.$$

The family of wavelets $\psi_{a,\Theta,\boldsymbol{b}}$ is called daughter Clifford-wavelet with $a \in \mathbb{R}^+$- dilation, Θ- rotation and $\boldsymbol{b} \in \mathbb{R}^{(3,1)}$- translation vector parameters.

3.1 MATLAB code: Plot of $\boldsymbol{\psi_{a,\Theta,b}(x)}$ in $\boldsymbol{Cl_{(3,1)}}$

```
sigma = 1;
omega_0 = 5;
t = linspace(-10, 10, 1000);
[X, Y] = meshgrid(t, t);


wavelet_scalar = exp(-X.^2 / (2 * sigma^2)) .* exp(1i * omega_0 * X);
wavelet_vector_x = exp(-X.^2 / (2 * sigma^2)) .* cos(omega_0 * X);
wavelet_vector_y = exp(-Y.^2 / (2 * sigma^2)) .* sin(omega_0 * Y);
```

```
wavelet_bivector = exp(-(X.^2 + Y.^2) / (2 * sigma^2)) .* ...
                                          (cos(omega_0 * X) .* sin(omega_0 * Y));

[X3, Y3, Z3] = meshgrid(t, t, linspace(-10, 10, 100));

wavelet_trivector = exp(-(X3.^2 + Y3.^2 + Z3.^2) / (2 * sigma^2)) .* ...
(cos(omega_0 * X3) .* sin(omega_0 * Y3) .* sin(omega_0 * Z3));
wavelet_quadvector = exp(-(X.^2 + Y.^2) / (2 * sigma^2)) .* ...
(cos(omega_0 * X) + sin(omega_0 * Y));

figure;

subplot(3, 2, 1);
plot(t, real(wavelet_scalar(500, :)), 'b', 'DisplayName', 'Real Part');
hold on;
plot(t, imag(wavelet_scalar(500, :)), 'r', 'DisplayName', 'Imaginary Part');
title('Scalar Wavelet');
xlabel('Time'); ylabel('Amplitude'); legend;

subplot(3, 2, 2);
plot(t, wavelet_vector_x(500, :), 'b', 'DisplayName', 'X Component');
hold on;
plot(t, wavelet_vector_y(:, 500), 'r', 'DisplayName', 'Y Component');
title('Vector Wavelet');
xlabel('Time'); ylabel('Amplitude'); legend;

subplot(3, 2, 3);
contourf(X, Y, wavelet_bivector, 20, 'LineColor', 'none');
colorbar;
title('Bivector Wavelet');
xlabel('X'); ylabel('Y');

subplot(3, 2, 4);
slice(X3, Y3, Z3, wavelet_trivector, 0, 0, 0);
title('Trivector Wavelet');
xlabel('X'); ylabel('Y'); zlabel('Z'); colorbar;

subplot(3, 2, 5);
plot(t, wavelet_quadvector(:, 500), 'b', 'DisplayName', 'Quadvector Wavelet');
title('Quadvector Wavelet');
xlabel('Time'); ylabel('Amplitude'); legend;
sgtitle('Clifford-Wavelet Function Components', 'Interpreter', 'none');
```

3.2 Plot of $\boldsymbol{\psi}_{\boldsymbol{a},\boldsymbol{\Theta},\mathbf{b}}(\mathbf{x})$

The graphs using Matlab code [10] shows components of Clifford-wavelet function (3.1) with scalar, vector, bivector, trivector, and quadra vector as displayed.

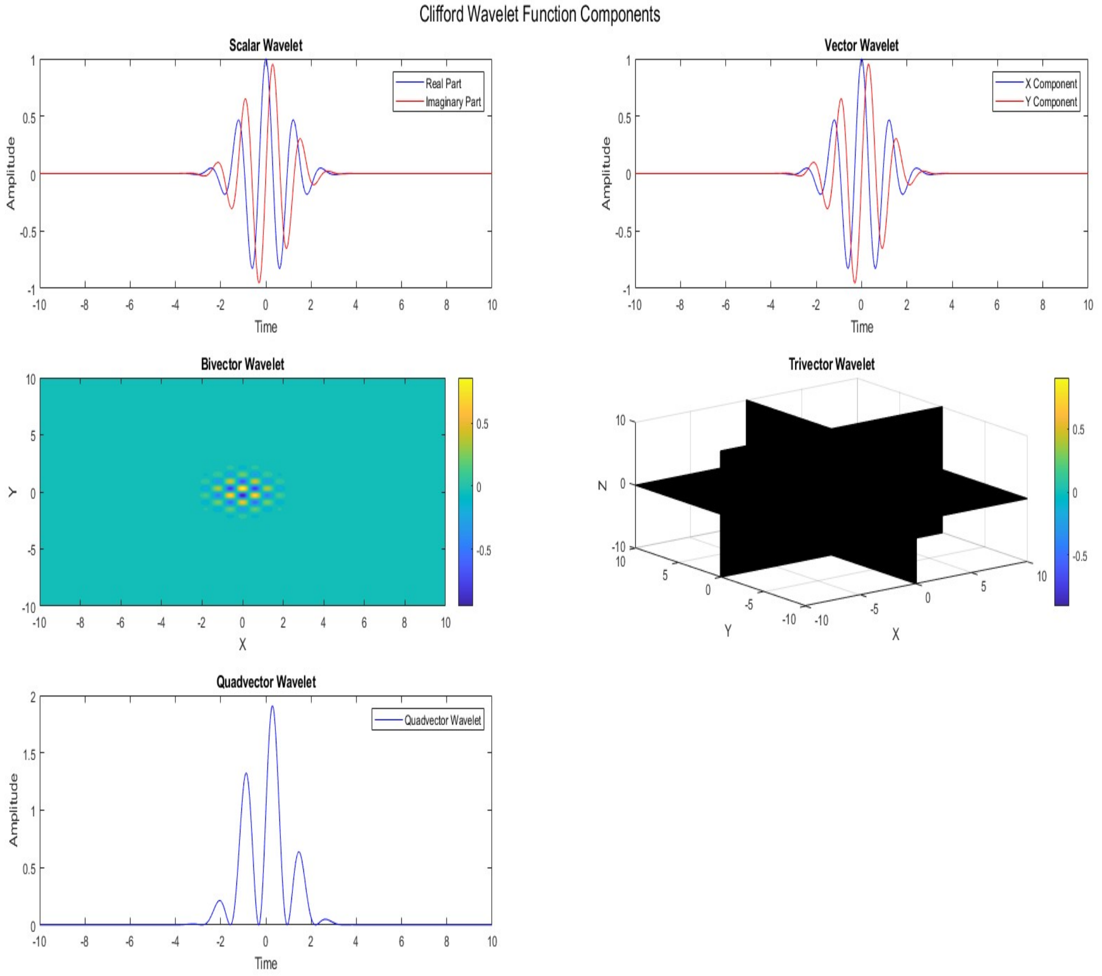


Figure 3.2: Plot of $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ considering equation (3.1)

3.3 Dirac Equation and Spinors in Clifford algebra

The Dirac equation is given as in [7]

$$(i\gamma^{\mu}\partial_{\mu}-m)\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})=0. \tag{3.2}$$

Various components of $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ in $Cl_{(3,1)}$ are shown below:

$$\begin{aligned}(i\gamma^{1}\partial_{1}-m)\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})&=0\\(i\gamma^{2}\partial_{2}-m)\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})&=0\\(i\gamma^{3}\partial_{3}-m)\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})&=0\\(i\gamma^{4}\partial_{4}-m)\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})&=0\end{aligned} \tag{3.3}$$

where γ^{μ} are the gamma matrices; ∂_{μ} is the four-gradient; m is the mass of the particle;

$\psi_{a,\Theta,b}$ is the spinor field (Clifford- wavelet function); i stands for the imaginary unit, which is defined as $i=\sqrt{-1}$; $\mu=1,2,3,4.$; $I_4=e_{1234}$.

3.4 Gamma Matrices

The gamma matrices of (3.2) γ^{μ} $(\mu=1,2,3,4)$ are a set of 4x4 matrices that satisfy the anticommutation relations [9, 11] similiar to our study of CA:

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}I_4 \tag{3.4}$$

where $M^{\mu\nu}$ is the metric tensor of space-time and I_4 is the 4x4 identity matrix.
The metric tensor in flat space-time in $Cl_{(3,1)}$ is given by:

$$M^{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}. \tag{3.5}$$

The standard representation (Dirac representation) of the gamma matrices in $Cl_{(3,1)}$ is given by:

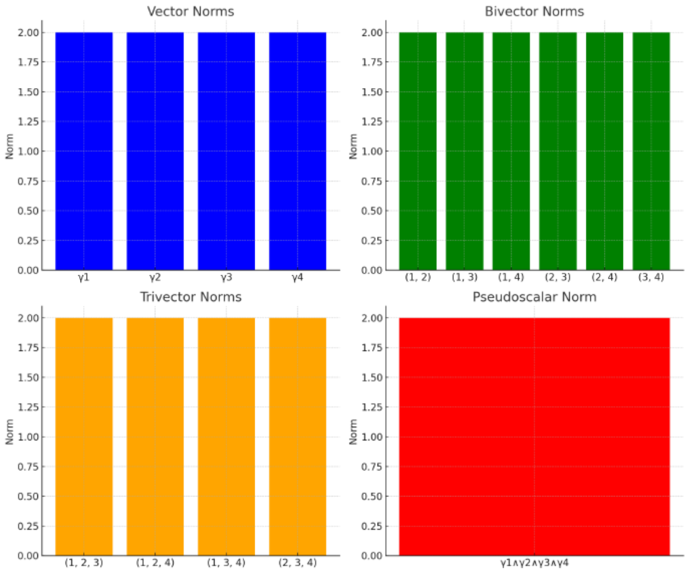
$$\gamma^1=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix},\quad \gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix}\quad (j=2,3,4) \tag{3.6}$$

where I_2 is the 2x2 identity matrix and σ^j are the Pauli matrices:

$$\sigma^2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma^4=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

The gamma matrices $\gamma^1,\gamma^2,\gamma^3,\gamma^4$ in this setup are 4×4 matrices that can be computed explicitly using the definitions of I_2 (the 2×2 identity matrix) and the Pauli matrices σ^j. To create plots for vectors, bivectors, trivectors, and quad-vectors using these gamma matrices, we interpret these as components in Clifford algebra.
The Gamma matrices construction explicitly can be represented as $\gamma^1,\gamma^2,\gamma^3,\gamma^4$. The vector representation single gamma matrices γ^{μ} ($\mu=1,2,3,4$) represent vectors. Bivector representation can be done using wedge products of two gamma matrices, e.g.,
$\gamma^{\mu}\wedge\gamma^{\nu}=\frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]$, represent bivectors and trivectors can be triple products like $\gamma^{\mu}\wedge\gamma^{\nu}\wedge\gamma^{\lambda}$. The quad-vector representation is given by the product of all four gamma matrices, e.g., $\gamma^1\wedge\gamma^2\wedge\gamma^3\wedge\gamma^4$, forms a pseudoscalar (quad-vector).

Constructing the matrices and interpreting in their respective spaces is shown in the plots. The visualizations for the norms of the vector, bivector, trivector, and pseudoscalar components derived from the gamma matrices is displayed as:


Vector Norms

Bivector Norms

Trivector Norms

Pseudoscalar Norm

Figure 3.4: Gamma matrices in $Cl_{(3,1)}$

3.5 Solution to the Dirac Equation

Definition 3.2 *Spinors are the solutions to the Dirac equation and can be represented as column vectors of four-component spinors as in (3.1). The components are complex-valued functions that describe the quantum state of a spin* $(-1/2)$ *particle.*
The Dirac equation is given by as in [8]:

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})=0. \tag{3.7}$$

On solving (3.7)

$$\psi_{a,\Theta,b}(\boldsymbol{x})=u(p)e^{-ip\cdot\boldsymbol{x}} \tag{3.8}$$

where $u(p)$ is a spinor and $p\cdot\boldsymbol{x}=p_{\mu}x^{\mu}=p^{0}\boldsymbol{x}^{0}-\vec{p}\cdot\vec{x}$.
Substituting (3.8) into (3.7)

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})=\left(i\gamma^{\mu}\partial_{\mu}-m\right)\left(u(p)e^{-ip\cdot\boldsymbol{x}}\right)=0. \tag{3.9}$$

Differentiating (3.8)

$$\partial_{\mu}\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})=\partial_{\mu}\left(u(p)e^{-ip\cdot\boldsymbol{x}}\right)=-ip_{\mu}u(p)e^{-ip\cdot\boldsymbol{x}}. \tag{3.10}$$

Substituting (3.10) into (3.7)

$$\left(i\gamma^{\mu}(-ip_{\mu})-m\right)u(p)e^{-ip\cdot\boldsymbol{x}}=0$$

$$\left(\gamma^{\mu}p_{\mu}-m\right)u(p)e^{-ip\cdot\boldsymbol{x}}=0.$$

Since $e^{-ip\cdot\boldsymbol{x}}$ is never zero hence one can factor it out:

$$\left(\gamma^{\mu}p_{\mu}-m\right)u(p)=0. \tag{3.11}$$

(3.11) is the algebraic form of the Dirac equation (3.7) to find the solutions for $u(p)$. The matrix $\left(\gamma^{\mu}p_{\mu}-m\right)$ must have a non-trivial kernel for non-zero solutions of $u(p)$. The solution is obtained in the form of eigenvalues and eigenvectors of the operator $\gamma^{\mu}p_{\mu}$. The eigenvalue problem for $\gamma^{1}$is a non-trivial solution of $u(p)$. The general solutions for $u(p)$

in different reference frames can be found by Lorentz transforming the rest of the solutions. Typically, solutions are found using the representation of the gamma matrices with a specific basis.

4 Transformation

In this section the detailed study of similitude group using rotors in $Cl_{(3,1)}$ is demonstrated using the definitions and developed theory from [9]. The hence developed wavelet function and spinors are used in various transformations to generalize the existing study in $Cl_{2,0}$.

4.1 Lorentz transformation

Lorentz transformations are essential in understanding space-time symmetries in the context of special relativity. The algebra of the Lorentz group can be described using the CA $Cl_{(3,1)}$.
A Lorentz transformation γ^{μ}_{α} and γ^{ν}_{β} preserves the Minkowski metric:

$$\gamma^{\mu}_{\alpha}\gamma^{\nu}{}_{\beta}\eta_{\mu\nu} = \eta_{\alpha\beta} \tag{4.1}$$

where $\alpha, \beta = 1,2,3,4$ and using (2.1).
In the context of CA, one can represent Lorentz transformation using rotors. A rotor R as defined in [9] and [1] is an element of the CA that satisfies: $\tilde{R}R = R\tilde{R} = 1$, where $\tilde{R}$ is the reverse of R. A Lorentz transformation of a vector $\boldsymbol{x} = \boldsymbol{x}^{\mu}e_{\mu}$ can be written as:
$\tilde{R}R = R\tilde{R} = 1, \boldsymbol{x}' = \tilde{R}\boldsymbol{x}R$.
The Lorentz group is represented on spinors using CA.
For an infinitesimal Lorentz transformation parameterized by $\omega^{\mu\nu}$, the transformation of a spinor $\psi_{a,\Theta,b}(\boldsymbol{x})$ as in (3.1) is:

$$\psi_{a,\Theta,b}(\boldsymbol{x}) \to \psi'_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) = \exp\left(-\tfrac{i}{4}\omega^{\mu\nu}\gamma^{\mu\nu}\right)\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) \tag{4.2}$$

where the generators $\gamma^{\mu\nu}$ are given (2.3).

4.2 Majorana spinors

Majorana spinors in $Cl_{(3,1)}$ are a particular type of spinor where the spinor field (CWF) is equal to its charge conjugate. This concept is naturally expressed within the framework of CA.
A Majorana spinor $\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})$ satisfies:

$$\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) = C\widetilde{\psi_{a,\Theta,\boldsymbol{b}}}(\boldsymbol{x})^{T} \tag{4.3}$$

where C is the charge conjugation matrix in CA.

4.3 Supersymmetry Algebra

Supersymmetry Algebra in $Cl_{(3,1)}$ extends space-time symmetries algebra by including supercharges Q_{α} and $\tilde{Q}^{\alpha}$, which are spinor generators that transform fermions into bosons and vice versa.
The Supersymmetry Algebra in $Cl_{(3,1)}$ can be written as:

$$Q_\alpha \tilde{Q}^\beta + \tilde{Q}^\beta Q_\alpha = 2\sigma^\mu_{\alpha\beta} P_\mu \tag{4.4}$$

where Q_α are the supersymmetry generators (fermionic operators); $\tilde{Q}^\alpha$ are their conjugate counterparts; P_μ are the generators of translations (momentum operators); σ^μ are the Pauli matrices related to the gamma matrices in the CA.

4.4 Supersymmetry Transformation

A supersymmetry transformation in $Cl_{(3,1)}$ involves the variation of a field under the action of a supercharges. For a scalar field $\phi(\boldsymbol{x})$, a fermionic field [9] $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$, and a gauge field $A_\mu(\boldsymbol{x})$, the Supersymmetry transformation can be written as:

$$\delta\phi(\boldsymbol{x}) = \tilde{\epsilon}\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) \tag{4.5}$$

$$\delta\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = -i\sigma^\mu \tilde{\epsilon}\partial_\mu \phi(x) + F(\boldsymbol{x})\epsilon \tag{4.6}$$

$$\delta A_\mu(\boldsymbol{x}) = \tilde{\epsilon}\sigma_\mu \lambda(\boldsymbol{x}) - \tilde{\lambda}(\boldsymbol{x})\sigma_\mu \epsilon \tag{4.7}$$

where ϵ is a spinor parameter of the Supersymmetry transformation; $F(\boldsymbol{x})$ is an auxiliary field; $\lambda(\boldsymbol{x})$ is a gaugino field associated with the gauge field $A_\mu(\boldsymbol{x})$. Supersymmetry transformations in $Cl_{(3,1)}$ are deeply connected to the structure of the CA, mainly through the use of gamma matrices and spinors. These transformations extend the symmetries of the space-time described by $Cl_{(3,1)}$ and play a central role in theories that attempt to unify bosonic and fermionic fields.

4.5 Gauge Transformation

In gauge theories, spinor fields transform under gauge transformation. For a gauge field A_μ and a spinor $\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})$, the covariant derivative is defined as:

$$D_\mu \psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) = (\partial_\mu - igA_\mu)\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}) \tag{4.8}$$

where g is the gauge coupling constant.

4.6 Chiral Symmetry

Chiral symmetry transformation in CWF is an example of left hand (LH) component of Dirac spinor [12] is defined as:

$$\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})_L = \tfrac{1}{2}(1 - \gamma^{1234})\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x}). \tag{4.9}$$

and right hand (RH) component of Dirac spinor is given by

$$\psi_{a,\Theta,\boldsymbol{b}}(x)_R = \tfrac{1}{2}(1 + \gamma^{1234})\psi_{a,\Theta,\boldsymbol{b}}(x) \tag{4.10}$$

where $\gamma^{1234} = i\gamma^1\gamma^2\gamma^3\gamma^4$.

Under a chiral transformation, the LH spinor component transform represented as

$$\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})_L \to e^{i\Theta}\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})_L. \tag{4.11}$$

and RH spinor component as

$$\psi_{a,\Theta,\boldsymbol{b}}(\boldsymbol{x})_R \to e^{-i\Theta}\psi_{a,\Theta,\boldsymbol{b}}(x)_R. \tag{4.12}$$

Conclusion

Authors have studied Dirac equation in Clifford algebra $Cl_{(3,1)}$ using the representation of the gamma matrices. Graphs related to Clifford wavelet function and program based on basic vectors are shown using MATLAB-R2022b. The Clifford wavelet function is applied to various transformations to observe the outcome using Clifford algebra $Cl_{(3,1)}$.

Future Work

Researchers in future can use the hence developed approach in the present study illustrated by the authors in Clifford algebra $Cl_{(3,1)}$ to establish Dirac equation with their spinors in quantum mechanics; in particular electromagnetism theory and its applications.

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Conflict of interest

The authors declare there is no conflicts of interest.

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