UtilitasMathematica

ISSN 0315-3681 Volume 122, 2025

Laguerre Expansions of C-resolvent for uniformly bounded β −times Integrated C −regularized semigroups Functions.

***Youssef Bajjou*[1], *Aziz Blali*[2] *and Abdelkhalek El Amrani*[3]**

[1,3]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University, Dhar El mahraz Faculty of Science, Fes, Morocco.

[2]Department of mathematics, Sidi Mohamed Ben Abdellah University, Higher normal school, Fes, Morocco.

youssefbajjou2017@gmail.com[1], aziz.blali@usmba.ac.ma[2], abdelkhalek.elamrani@usmba.ac.ma[3]

Abstract

The aim of this paper is to approximate the C-resolvent of uniformly bounded β −times integrated C −regularized semigroups function by the Laguerre series, recalling the notions and the results used.

AMS Mathematics Subject Classification (2020): Primary 47D62,47D60, 20M75, 33C45, A7A58.

Keywords and phrases: *integrated semigroups, C-semigroups, Generalisations of semigroups, orthogonal polynomials, linear operator approximation theory.*

1. Introduction and preliminaries

One of the important tools in mathematical physics is the series expansion of Laguerre orthogonal polynomials, for example in problems involving the integration of Helmholtz's equation in prabolic coordinates, in the theory of the Hydrogen atom, in the theory of propagation of electomagnetic waves a long transmission lines terminated by a lumped inductance [9]. The study of sufficient conditions for the convergence of Laguerre series has been the subject of numerous works, for more details see [16], [4], [15], [9] and [2].

In 2023, Y.Bajjou, A. Blali and A. El Amrani in their article [1], studie the Laguerre Expansions of exponentially bounded C-regularized semigroups functions and its C-resolvent. In this work we will be interested in Laguerre expansion of the C-resolvent of uniformly bounded $\beta-$ times Integrated C −regularized semigroups Functions, starting with reminding the notations, concepts and results used. Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E,F)$ denotes the set of all applications from E to another Banach space F, $B(E)$ denotes the space of all bounded linear operators from E into itself, and $L^1_{loc}(E)$ the set of all $f\in\mathfrak{F}(\mathbb{R},E)$ locally integrable. For a closed linear operator A on E, $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\parallel x\parallel_{\mathcal{D}(A)}=\parallel x\parallel_E+\parallel Ax\parallel_E$ become Banach space. Throughout this paper, $C\in B(E)$ will be an injective operator. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A):=\{\lambda\in\mathbb{C}|R(C)\subseteq R(\lambda I-A)\ and\ \lambda I-A$ is injective in $B(E)\}$ and if $\lambda\in\rho_C(A)$ then we denoted by $R_C(\lambda,A)=$

$(\lambda I-A)^{-1}C$ the C-resolvent. Finally we note for $a,b\in\mathbb{R}^{+}$ such that $a\geqslant b$, $\lfloor a\rfloor$ the integer part of real number a, $a!=\Gamma(a+1)$ and $\binom{a}{b}=\frac{a!}{b!(a-b)!}=\frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$.

Laguerre functions and Laguerre expansions on Banach spaces

For all $n\in\mathbb{N}$, and arbitrary real $\alpha>-1$ the classical Laguerre polynomial, is defined by Rodrigues formula:

$$(\forall x\in\mathbb{R})\ \phi_{n,\alpha}(x)=e^{x}\frac{x^{-\alpha}}{n!}\frac{d^{n}}{dx^{n}}(e^{-x}x^{n+\alpha})$$

. $\phi_{n,\alpha}$ is a polynomial with the degree n, the same parity as n, whose highest monomial degree is $\frac{(-1)^{n}}{n!}X^{n}$ and have real coefficients. Furthemore, they verify the following condition of orthogonality wiht respect to gamma density $x\to x^{\alpha}e^{-x}$ on $[0,+\infty[$:

$$\int_{\mathbb{R}^{+}}\phi_{n,\alpha}(x)\phi_{m,\alpha}(x)x^{\alpha}e^{-x}dx=\delta_{n,m}\frac{\Gamma(n+\alpha+1)}{n!},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and satisfied the estimates:

$$(\forall x\in\mathbb{R}^{+})\ (\exists c_{x}>0)\ (\exists n_{0}\in\mathbb{N})\ (\forall n\geq n_{0}),|\phi_{n,\alpha}(x)|\leq c_{x}n^{\frac{\alpha}{2}}. \tag{1}$$

For more details of the classical theory of orthogonal polynomials see [9], [16], [4], [15], [1] and [2].

The Laguerre functions on $\mathbb{R}^{+}$ are defined by:

$$\varphi_{n,\alpha}(x)=\sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\phi_{n,\alpha}(x)x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}=\frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}}x^{\frac{-\alpha}{2}}e^{\frac{x}{2}}\frac{d^{n}}{dx^{n}}(e^{-x}x^{n+\alpha}). \tag{2}$$

$(\varphi_{n,\alpha})_{n\in\mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^{2}(\mathbb{R}^{+})$ and satisfied some recurrence relations, equality and inequality. For more details see [16], [4], [1], [15] and [2].

For $n\in\mathbb{N}$ and arbitrary real $\alpha>-1$, we denote by $\psi_{n,\alpha}$, the function on $\mathbb{R}^{+}$ defined by :

$$(\forall x\in\mathbb{R}^{+})\ \psi_{n,\alpha}(x)=\frac{n!}{\Gamma(n+\alpha+1)}x^{\alpha}e^{-x}\phi_{n,\alpha}(x)=\frac{1}{\Gamma(n+\alpha+1)}\frac{d^{n}}{dx^{n}}(x^{n+\alpha}e^{-x}). \tag{3}$$

$\psi_{n,\alpha}$ satisfies recurrence relations and differential equations, for example:

$$(\forall(n,m)\in\mathbb{N}^{2})\ \psi_{n,\alpha}^{(m)}=\psi_{n+m,\alpha-m}. \tag{4}$$

And the following useful inequality :

$$(\forall n\geq 1)\quad \|\psi_{n,\alpha}\|_{1}\leq\frac{C_{\alpha}}{n^{\frac{\alpha}{2}}}. \tag{5}$$

For more details see [1] and [9].

Of the most important properties of the family $(\psi_{n,\alpha})_{n\in\mathbb{N}}$ is that if $f:\mathbb{R}_{*}^{+}\to E$ be a differentiable function such that $\int_{0}^{+\infty}e^{-t}t^{\alpha}\|f(t)\|^{2}dt<+\infty$, then the series $\sum_{n\in\mathbb{N}}c_{n}(f)\phi_{n,\alpha}(t)$ converges pointwise to f on $\mathbb{R}_{+}^{*}$, where

$$c_{n}(f)=\int_{0}^{+\infty}\psi_{n,\alpha}(t)f(t)dt.$$

For more details see [9], [4] and [15].

$\boldsymbol{\beta}-$ times integrated $\boldsymbol{C}-$regularized semigroups

Let $\beta\geq 0$. A strongly continuous family $(T(t))_{t\geq 0}$ in $B(E)$ is called $\beta-$times uniformly bounded integrated $C-$regularized semigroups generated by W, or uniformly bounded $\beta-$ times Integrated $C-$semigroups generated by W, if

1. For all $x \in E$, $T(0)x = \begin{cases} Cx & if \ \ \beta = 0, \\ 0 & otherwise. \end{cases}$.
2. There exists $M \geq 0$ such that:
$$(0, +\infty) \subset \rho_C(W), \ \| T(t) \| \leq M, for\ all\ t \geq 0.$$
3. For $\lambda > 0$ and $x \in E$, we have $R_C(\lambda, W)x := (\lambda I - W)^{-1}Cx = \lambda^\beta \int_0^{+\infty} e^{-\lambda t} T(t)x dt$.

We can deduce from [11] that,

4. For any $x \in E$,
$$T(t)T(s)x = \frac{1}{\Gamma(\beta)}\left[\int_t^{t+s} (t+s-r)^{\beta-1} T(r)Cxdr - \int_0^s (t+s-r)^{\beta-1} T(r)Cxdr\right] \ for\ all\ t, s \in \mathbb{R}^+. \quad (6)$$
5. For all $t \geq 0$, $T(t)C = CT(t)$

We present some known facts about $\beta-$ times integrated exponentially bounded C-semigroups and its generator, which will be used in the sequel (see [6], [7], [8], [10], [12], [13], [14], [17] and [18] for more details):

• By the equation (6), we conclude that $T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$, this means that $T(t)x \in \mathcal{D}(W)$, $WT(t)x = T(t)Wx$ and $T(t)x = \frac{t^\beta}{\Gamma(\beta+1)}Cx + \int_0^t T(s)Wxds$, for all $t \geq 0$ and $x \in D(W)$.

• $\int_0^t T(s)xds \in \mathcal{D}(W)$ and
$$W\int_0^t T(s)xds = T(t)x - \frac{t^\beta}{\Gamma(\beta+1)}Cx \ \ for\ every\ x \in E\ and\ t \geq 0 \quad (7)$$
, which implies that for each $x \in \mathcal{D}(W)$, $u := T(.)x$ is right differentiable in $t \geq 0$ and $\frac{d}{dt}T(t)x = WT(t)x + \frac{t^{\beta-1}}{\Gamma(\beta)}Cx$.

• W is closed linear operator with $R(C) \subset \overline{D(W)}$ and $W = C^{-1}WC$.

• The C-resolvent operator $R_C(\lambda, W)$ is analytic in the C-resolvent set $\rho_C(W)$ and
$$\frac{d^n}{d\lambda^n}(R_C(\lambda, W)) = (-1)^n n!\,(R_C(\lambda, W))^{n+1} foralln \in \mathbb{N}. \quad (8)$$

• Like $\forall\ \gamma > 0\ \ \forall\ \lambda > 0\ \ \int_0^{+\infty} t^{\gamma-1}e^{-\lambda t}dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ then,

- For every $n \in \mathbb{N}$, $R(C) \subset D((\lambda I - W)^{-n})$ and
$$(\lambda I - W)^{-n}Cx = \frac{\lambda^\beta}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)xd. \quad (9)$$
which gives $\| (\lambda I - W)^{-n}C \| \leq \lambda^{\beta-n}M$.

- Let $\gamma > 0$ and $\lambda > 0$.

We can define the fractional power of C-resolvent operator(see [6] for more details) as below :

$$(\lambda I-W)^{-\gamma}Cx=(\lambda I-W)^{-\gamma}Cx:=\frac{\lambda^{\beta}}{\Gamma(\gamma)}\int_0^{+\infty} t^{\gamma-1}e^{-\lambda t}T(t)xdt, for\ all\ x\in E. \quad (10)$$

With a simple verification, $(\lambda I-W)^{-\gamma}C$ is bounded linear operator. Therefore we can deduce that for $x\in E$ fixed and $k\in\mathbb{N}$,

$$\| (R_C(\lambda,W))^k x \|_E \leq M^k\lambda^{k(\beta-1)} \| x \|_E \quad (11)$$

- From equations (8) and inequation (11) we can easily conclude that

$$\forall\alpha>0,\ \forall n\in\mathbb{N},\ \forall k\in\{0,..,n\},\ [\frac{d^{n-k}}{dt^{n-k}}(e^{-t}t^{n+\alpha})\frac{d^k}{dt^k}(R_C(t,W))]_0^{+\infty}=0 \quad (12)$$

2. Main results

Lemma 1 *Let $\beta\geqslant 0$ and $(T(t))_{t\in\mathbb{R}^+}$ be an $\beta-$ times integrated uniformly bounded C-semigroups in Banach space E with generator $(W,\mathcal{D}(W))$ such that* $(\forall t\geq 0):\ \| T(t) \|\leq M$. For any $n\in\mathbb{N}$, $x\in E$ and $\alpha>1$, we have

$$\int_0^{+\infty}\psi_{n,\alpha}(t)R_C(t,W)xdt=\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)}\int_0^{+\infty}\frac{s^n}{(s+1)^{n+\alpha+\beta+1}}T(s)xds$$

Proof: Let $\beta\geqslant 0$ and $(T(t))_{t\in\mathbb{R}^+}$ be an $\beta-$ times integrated uniformly bounded C-semigroups in Banach space E with generator $(W,\mathcal{D}(W))$, we have, For all $t>0$, $t\in\rho_C(W)$, so the function $t\to R_C(t,W)$ is analytic in $\mathbb{R}_*^+$.

Or by the inequality (11), we know that $\| R_C(t,W) \|\leq Mt^{\beta-1}$ for all $t>0$, so

$$\int_0^{+\infty} \| \psi_{n,\alpha}(t)R_C(t,W)x \| dt=\int_0^{+\infty}|\psi_{n,\alpha}(t)| \| R_C(t,W)x \| dt$$
$$\leq\int_0^{+\infty}\frac{n!Mt^{\beta-1}}{\Gamma(n+\alpha+1)}e^{-t}t^{\alpha}|\phi_{n,\alpha}(t)| \| x \| dt$$
$$\leq\frac{n!M\|x\|}{\Gamma(n+\alpha+1)}\int_0^{+\infty} e^{-t}t^{\alpha+\beta-1}|\phi_{n,\alpha}(t)|dt$$
$$<+\infty \quad (because\ \alpha+\beta-1\geqslant 0).$$

If we posed $H:=\Gamma(n+\alpha+1)\int_0^{+\infty}\psi_{n,\alpha}(t)R_C(t,W-\omega I)xdt$, then we have

$$H=\Gamma(n+\alpha+1)\int_0^{+\infty}\psi_{n,\alpha}(t)R_C(t,W)xdt$$
$$=\int_0^{+\infty}\frac{d^n}{dt^n}(e^{-t}t^{n+\alpha})R_C(t,W)xdt$$
$$=[\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^{n+\alpha})R_C(t,W)]_0^{+\infty}-\int_0^{+\infty}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^{n+\alpha}\frac{d}{dt}(R_C(t,W)x)dt$$
$$=0-[\frac{d^{n-2}}{dt^{n-2}}(e^{-t}t^{n+\alpha})\frac{d}{dt}(R_C(t,W)x)]_0^{+\infty}+$$
$$(-1)^2\int_0^{+\infty}\frac{d^{n-2}}{dt^{n-2}}(e^{-t}t^{n+\alpha}\frac{d^2}{dt^2}(R_C(t,W)x)dt$$
$$=(-1)^n\int_0^{+\infty}e^{-t}t^{n+\alpha}\frac{d^n}{dt^n}(R_C(t,W))dt\ (integtration\ by\ parts)$$
$$=(-1)^n\int_0^{+\infty}e^{-t}t^{n+\alpha}(-1)^n n!\,R_C(t,W)^{n+1}xdt$$
$$=\int_0^{+\infty}e^{-t}t^{n+\alpha}(n!)\frac{t^{\beta}}{n!}\int_0^{+\infty}s^ne^{-ts}T(s)xdsdt$$
$$=\int_0^{+\infty}s^nT(s)x\{\int_0^{+\infty}t^{n+\alpha+\beta}e^{-t}e^{-ts}dt\}ds\ (Fubini's\ Theorem)$$

$$= \int_0^{+\infty} s^n T(s)x\{\int_0^{+\infty} t^{n+\alpha+\beta} e^{-(s+1)t} dt\} ds$$
$$= \int_0^{+\infty} s^n T(s)x\{\int_0^{+\infty} \frac{u^{n+\alpha+\beta}}{(s+1)^{n+\alpha+\beta}} e^{-u} \frac{du}{s+1}\} ds \ \ (u = (s+1)t)$$
$$= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x\{\int_0^{+\infty} u^{n+\alpha+\beta} e^{-u} du\} ds$$
$$= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x\Gamma(n+\alpha+\beta+1) ds$$
$$= \Gamma(n+\alpha+\beta+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x ds$$

hence the result.

Theorem 2.1 *Let $\beta \geqslant 0$ and $(T(t))_{t\in\mathbb{R}^+}$ be an $\beta -$ times integrated uniformly bounded C-semigroups in Banach space E with generator $(W, \mathcal{D}(W))$. For $x \in \mathcal{D}(W)$ and $\alpha > 1$, we have :*

$$R_C(t, W)x = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)} \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x ds \ \ \phi_{n,\alpha}(t)$$

Proof : Throughout the proof, α is an arbitrary real such that $\alpha > 1$ and $x \in \mathcal{D}(W)$. The function $t \to R_C(t, W)x$ is differentiable in $\mathbb{R}_*^+$ (because she is analytic in $\mathbb{R}_*^+$), and

$$\int_0^{+\infty} t^\alpha e^{-t} \parallel R_C(t, W)x \parallel^2 = \int_0^{+\infty} t^\alpha e^{-t} \parallel (tI - W)^{-1} Cx \parallel^2$$
$$\leq \int_0^{+\infty} t^\alpha e^{-t} M^2 t^{2\beta-2} \parallel x \parallel^2 dt$$
$$\leq M^2 \parallel x \parallel \int_0^{+\infty} t^{\alpha+2\beta-2} e^{-t} dt$$
$$\leq M^2 \parallel x \parallel \Gamma(\alpha + 2\beta - 1) \ (because \ \ \alpha > 1 \ \ and \ \ \beta \geqslant 0)$$
$$< +\infty.$$

Thus, the series $\sum_{n\in\mathbb{N}} c_n(R_C(.,W)x)\phi_{n,\alpha}$ converges pointwise to $R_C(.,W)x$ on $\mathbb{R}_+^*$, where

$$c_n(R_C(.,W)x) = \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W)x dt$$
$$= \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)} \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x ds \ \ (according \ \ of \ \ lemma \ \ 1).$$

Therefore

$$(\forall t > 0), \ \ R_C(t, W)x = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)} \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+\beta+1}} T(s)x ds \ \ \phi_{n,\alpha}(t).$$

Corollary 2.2 *Let m be belongs to $\mathbb{N}$. $(T(t))_{t\in\mathbb{R}^+}$ be a $m -$ times integrated uniformly bounded C-semigroups in Banach space E with generator $(W, \mathcal{D}(W))$, then For $x \in \mathcal{D}(W)$ and $\alpha > 1$, we have :*

$$\forall t > 0, \qquad R_C(t, W)x = m! \sum_{n=0}^{+\infty} \binom{\alpha+n+m}{\alpha+n} \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+m+1}} T(s)x ds \ \ \phi_{n,\alpha}(t)$$

Proof:

Like $\frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)} = m! \binom{\alpha+n+m}{\alpha+n}$ then the proof follows directly from theorem 2.1.

Example 2.3 *The space $X = \ell_1 = \{(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ | \ \sum_{n=0}^{+\infty} |x_k| < +\infty\}$, equipped* *whit the norm* $\| (x_k)_{k\in\mathbb{N}} \|_1 = \sum_{k=0}^{+\infty} |x_k|$ *becomes a Banach space. For each* $n \in \mathbb{N}$ *let* $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$ *be element of* X*. Like for all* $x = (x_k)_{k\in\mathbb{N}} \in X, x = \sum_{k=0}^{+\infty} x_k e_k$*, then* $X = span\{e_n |\ n \in \mathbb{N}\}$*. Considering the family of operators* $(T(t))_{t\geq 0}$ *defined by :*

$$\forall t \geqslant 0,\ \forall x = (x_k)_{k\in\mathbb{N}} \in X,\ T(t)x = \sum_{i=0}^{+\infty} \frac{(1-e^{-t})}{2} \delta_{i,2\lfloor \frac{i}{2} \rfloor} x_i e_i$$

and the operator C defined on X by :

$$\forall x = (x_k)_{k\in\mathbb{N}} \in X,\ Cx = \sum_{i=0}^{+\infty} \frac{1}{2} \delta_{i,2\lfloor \frac{i}{2} \rfloor} x_i e_i$$

$(T(t))_{t\geq 0}$ is 1-times integrated C-regularized semigroups, uniformly bounded ($\| T(t) \| \leq 1$) wiht generators $(W, \mathcal{D}(W))$ such that

$\mathcal{D}(W) = \{(x_i)_{i\in\mathbb{N}} \in X \mid \forall i \in 2\mathbb{N},\ x_i = 0\}$ and $\forall x \in \mathcal{D}(W),\ Wx = -\frac{1}{2}x$.

Theorem 2.1 give for $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{D}(W)$ fixed, $\forall t > 0$,

$$\begin{aligned} R_C(t,W)x &= 1! \sum_{n=0}^{+\infty} \binom{\alpha+n+1}{\alpha+n} \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1+1}} T(s)x ds\ \ \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (\alpha+n+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+2}} T(s)x ds\ \ \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (\alpha+n+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+2}} \sum_{i=0}^{+\infty} \frac{(1-e^{-s})}{2} \delta_{i,2\lfloor \frac{i}{2} \rfloor} x_i e_i ds\ \ \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} \sum_{i=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n (\frac{(1-e^{-s})}{2})}{(s+1)^{n+\alpha+2}} ds \right\}\ \delta_{i,2\lfloor \frac{i}{2} \rfloor}\ (\alpha+n+1) x_i e_i\ \ \phi_{n,\alpha}(t) \end{aligned}$$

Example 2.4 *Let* $m: \mathbb{R} \to \mathbb{R}^-$ *be an even mesurable function. In the Banach space* $L^1(\mathbb{R})$*, we consider the family* $T := (T(t))_{t\geq 0} \subset \mathfrak{F}(L^1(\mathbb{R}))$ *defined by* $\forall t \geq 0, T(t): L^1(\mathbb{R}) \to L^1(\mathbb{R}), f \mapsto T(t)(f): \mathbb{R} \to \mathbb{R}, s \mapsto T(t)(f)(s) = e^{t.m(s)} f(-s)$*. Clearly,* $(T(t))_{t\geq 0} \subset B(L^1(\mathbb{R}))$*. If we put,* $T(0) = C$*, then, a family of operators* $(T(t))_{t\geq 0}$ *is uniformly bounded 0-times integrated* C*-regularized semigroup. with generator* $(W, \mathcal{D}(W))$ *defined by*

$W: \mathcal{D}(W) = \{f \in L^1(\mathbb{R}) |\ m.f \in L^1(\mathbb{R})\} \to L^1(\mathbb{R}), f \mapsto W(f) = m.f$. Theorem 2.1 give for $f \in \mathcal{D}(W)$ fixed,

$$\forall t \in \mathbb{R}_*^+, R_C(t,W)(f) = \sum_{n=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{sm(.)} ds \right\} C(f) \phi_{n,\alpha}(t).$$

Example 2.5 *The space* $X = c_0 = \left\{ (x_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \lim_{k\mapsto+\infty} x_k = 0 \right\}$*, equipped wiht the norm* $\| (x_k)_{k\in\mathbb{N}} \|_\infty = max_{k\in\mathbb{N}} |x_k|$ *becomes a banach space. For each* $n \in \mathbb{N}$*, let* $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$ *be element of* X*. Since for all* $x = (x_k)_{k\in\mathbb{N}} \in X, x = \sum_{k=0}^{+\infty} x_k e_k$ *we have* $X = span\{e_n |\ n \in \mathbb{N}\}$*. Considering the family of operators* $(T(t))_{t\geq 0}$ *defined by :*

$$\forall t \in \mathbb{R}^+\ \forall x = \sum_{k=0}^{+\infty} x_k e_k \in X, T(t)x = \sum_{k=0}^{+\infty} e^{-k^2 t} x_k e_k$$

$(T(t))_{t\geq 0}$ is a unifomly bounded 0-times integrated I-semigroup ($\forall t \geqslant 0,\ \| T(t) \| \leq 1$) with generator $(W, \mathcal{D}(W))$ such that $\mathcal{D}(W) = \{x = (x_k)_{k\in\mathbb{N}} \in X / (k^2 x_k)_{k\in\mathbb{N}} \in X\}$ and $\forall x \in \mathcal{D}(W),\ Wx = \sum_{k=0}^{+\infty} - k^2 x_k e_k$. Theorem 2.1 give for $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{D}(W)$ fixed,

$$\forall t > 0,\ R_C(t,W)x = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-k^2 s} ds \right\} x_k e_k \phi_{n,\alpha}(t).$$

References

[1] L. Abadias And P. J. Miana, $C_0 -$ *semigroups and Resolvent operations*

approximated by Laguerre Expansion, ArXiv:1311. 7542v2 [math. FA] 32 oct 2014.

[2] R. Askey, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87, 695-708 (1965).

[3] Y. Bajjou, A. Blali And A. El. Amrani, *Laguerre Expansions of* C *−Semigroups Functions*, Siberian Electronic Mathematical Reports, DOI 10.33048 / semi. 2023.20.061, Tom 20, N2, cmp. 101-1012 (2023).

[4] R. Delaubenfels, C*-semigroups and the Cauchy problem*, J. Funct. Anal. 111 (1993), 44-61.

[5] R. Delaubenfels, *Existence families, functional calculi and evolution equations*, Lecture Notes in Math, 1570, Springer-Verlag(1994).

[6] R. O. Fattorini, *A note on fractional derivatives of semigroups and cosine functions*, Pacific journal of mathematics, Vol. 109, No 2, 1983.

[7] M. Kaltenback And F. L. Schwenninger, *Generalisations of Semigroups of Operators in the View of Linear Relations*, Analysis und Scientic Computing der Technischen Universitat Wien, Matr. Nr. 0625539 Wahringer Strasse 125/15-1180 Wien. Vienna University of Technology.

[8] M. Kostic, (a, k) *−Regularized* C *−Resolvent Families: Regularity and Local Properties*, Hindawi Publishing Corporation Abstract and Applied Analysis, Volume 2009, Article ID 858242, 27 pages doi:10. 1155/2009/858242.

[9] N. N. Lebedev, *Special function and their applications,* Prentice-hall, INC(1965).

[10] M. Li, Q. Zheng And J. Zhang, *Regulrized resolvent famillies*, Taiwanese Journal of Mathematics, vol.11, no.1, p. 117-133 (2007).

[11] M. Liu, D. Q. Liao, Q. Q. Zhu And F. H. Wang $\alpha -$ *Times Integrated* C *−Semigroups*, Advences in Pure Mathematics, 2012, 2, 211-215, http://dx.doi.org/10.4236/apm.2012.23030 published Online May 2012.

[12] C. Martinez, M. Sanz And L. Marco, *Fractional powers of operators*, J. Math. Soc. Japan, Vol. 40, No. 2, 1988.

[13] M. Mosallanezhada And M. Janfadab, *On Mixed* C *−Semigroups of Operators on Banach Spaces*, Filomat 30:10, 2673-2682 (2016).

[14] P. Preda, A. Pogan And C. Preda, *The Perron Problem For* C *−semigroups*, Math. J. Okayama Univ. 46, 141-151 (2004).

[15] J. A. Shohat, *on the developement of functions in series of orthogonal polynomials*, The university of Pennsyvania (1935).

[16] G. Szego, *Orthogonal polynomials*, PubLished by the American Mathematical Society Providence, Rhode Island (1939).

[17] Q. Zheng, *Controllability of a class of linear systems in Banach spaces*, Proc. Amer. Math. Soc. 123(1995), 1241-1251. MR 95e:93011. Analysis und Scientific Computing der Technischen Universitat Wien.

[18] Q. Zheng And Y. Zhao, *An analyticity criterion for regularezed semigroups*, proceedings of the american Mathematical society, Volume 130, Number 8, S0002-9939(02)06219-6 (2002).