b –Coloring of Comb Product of Extended Duplicate graph of Path Networks

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Abstract

The b – chromatic number which is $\varphi(G)$, the highest k such that graph can be b – colored using k colors. At least one vertex in every color group is adjacent to a vertex in every other color group. In this work we acquire b – chromatic number of the middle graph of an extension of duplicate graph of Path graph ($M[EDG(P_n)]$), total graph of an extension of duplicate graph of Path graph ($T[EDG(P_n)]$), Comb Product of middle graph of an extension of duplicate graph of path with (P_n) and Comb Product of total graph of an extension of duplicate graph of path with (P_n).

***Keywords* -** b-coloring, middle graph, extended duplicate graph, comb product, path.

1. Introduction

The coloring of a vertices should not be the same on two adjacent vertices. In this study, Graphs are finite, undirected, and have no loops or multiple edges, and color refers to vertex coloring.

b-coloring is the process of colouring a graph G's vertices so that at least one vertex in each colour group has a neighbour in every other colour group. In G, the largest integer k that permits b-coloring is called a b-chromatic number. In [2], a number of intriguing colouring concepts and associated parameters are examined.

Manlove and Irving proposed the concept of b-coloring, and it has been shown that the problem of determining the b-chromatic number is NP-hard[4].

The upper bound of $\varphi(G)$ has also been established by Irving and Manlove, which is $\varphi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the highest degree of G[10].

A *duplicate graph of G*, DG = (V^1, E^1) and the vertex set V^1 is the union of V and V' also $V \cap V' = \phi$. The function f from V to V' is bijective. The edge set E^1 of a duplicate graph has the edge u_1u_2' and $u_1'u_2$ if u_1u_2 is in E. The *extension of duplicate graph* of a duplicate graph, denoted by EDG, is defined as, any two vertices can be connected by an edge from V to any other vertex in V', except the end vertices. For convenience, we take $u_2 \in V$ and $u_2' \in V'$ and thus the edge u_2u_2' is formed[6,7,9].

If two vertices, x and y, represent neighbouring edges in G and are incident in G as well, then they are linked. The middle graph, M(G), is one in which the vertex set of M is the union of V(G) and E(G) [5].

The graph's complete graph T(G) The vertices and edges of a graph G are included in its vertex set. If the matching elements of two vertices in T are incident or nearby in G, then the vertices are related [5].

Let G and H represent two connected graphs. Let o be a vertex of H. The *comb product* between graphs G and H, denoted by $G \rhd H$, , is created by incorporating one instance of G along with |V(G)| instances of H. The i-th instance of H is connected to the i-th vertex of G at vertex o. The number of vertices in $G \rhd H$ is |V(G)|.|V(H)| and the number of edges is |V(G)|.|E(H)|+|E(G)|[1,8].

2. b-chromatic number of the middle graph of an extension of duplicate graph of path $(M[EDG(P_n)])$

The length of a path is the number of edges it has visited, and a path is a series of successive edges in a graph. Pn represents a route with n vertices. EDG(Pn) with 2n vertices and 2n-1 edges is the extension of a duplication graph of a route graph [7].

Theorem 2.1

An extension of a path graph's duplicate graph's middle graph's b-chromatic number is, $\varphi(M[EDG(P_n)]) = 5, n \geq 4$

Proof:

There are 4n-1 vertices and 6n-2 edges with a maximum degree of 6 and a minimum of 1 in the middle graph of an extension of the duplicate graph of path Pn. Four vertices v_1, v_n', v_1' and v_n have degree 1, 2n-6 vertices $v_i, v_i' : 3 \leq i \leq n-1$ have degree 2, four vertices v_2, v_2', x_n, and x_{2n} have degree 3, two vertices x_2 and x_{n+1} have degree 5, one vertex x_{2n+1} have degree 6 and the remaining vertices have degree 4.

Since $\Delta(M[EDG(P_n)]) = 6$, by the upper bound of $\varphi(G)$, $\varphi(M[EDG(P_n)]) \leq 7$.

Case (i):

If $\varphi(M[EDG(P_n)]) = 7$, then $M[EDG(P_n)]$ must have seven vertices of degree 6, which is unfeasible, as $M[EDG(P_n)]$ has only one vertex of degree 6. Consequently, $\varphi(M[EDG(P_n)]) \neq 7$.

Case(ii):

If $\varphi(M[EDG(P_n)]) = 6$ then $M[EDG(P_n)]$ must have six vertices of degree 5, which is also unfeasible, as $M[EDG(P_n)]$ has only two vertices of degree 5. Consequently, $\varphi(M[EDG(P_n)]) \neq 6$.

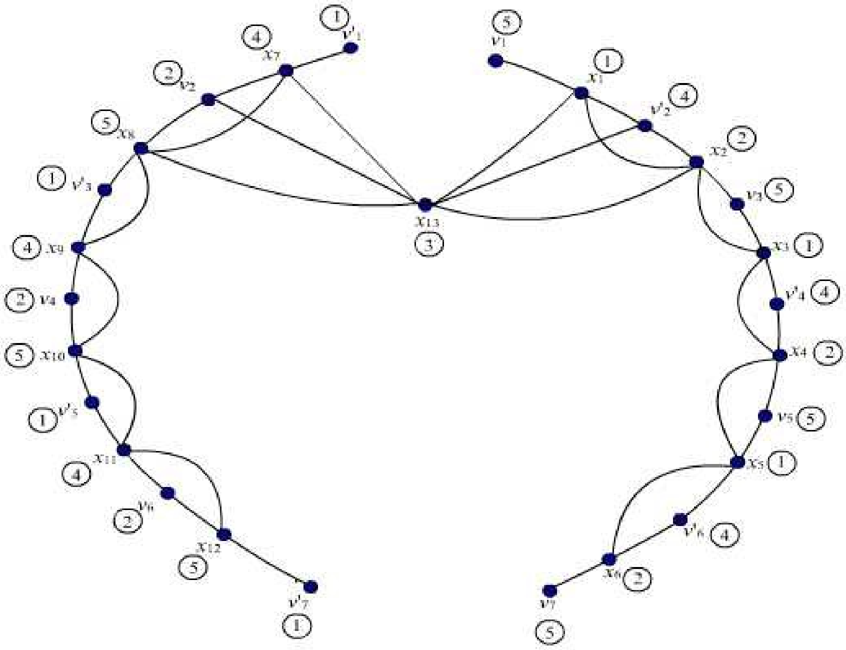

Figure 1 $\boldsymbol{M[EDG(P_7)}$

Case(iii):

Due to adjacency of vertices in $M[EDG(P_n)]$ maximum of five b- vertices can be generated for the proper coloring. Consider the following 5-coloring (c_1, c_2, c_3, c_4, c_5) of $M[EDG(P_n)]$, assign the color c_1 to x_1 , x_3 , v_1', assign the color c_2 to x_2 , v_2 , assign the color c_3 to x_{2n-1}, assign the color c_4 to v_2', x_n assign the color c_5 to v_1 , x_{n+1} and the remaining vertices can be assigned from one of the allowed color - such color exists. The b - vertices x_1, x_2, x_{2n-1}, x_n *and* x_{n+1} for the color classes c_1, c_2, c_3, c_4 *and* c_5 respectively produces the b - chromatic coloring.

Hence $\varphi(M[EDG(P_n)]) = 5\ , n \geq 4$

3. b-chromatic number of total graph of an extension of duplicate graph of path $(\boldsymbol{T[EDG(P_n)]}$

Theorem 3.1

An extension of a path graph's duplicate graph's total graph's b-chromatic number is, $\varphi(T[EDG(P_n)]) = 5\ , n \geq 4$

Proof:

With a maximum degree of 6 and a minimum of 2, the overall graph of an extension of a duplicate graph of path Pn consists of 4n-1 vertices and 8n-3 edges. It has four degree 2 vertices, two degree 3 vertices, fourn-12 degree 4 vertices, two degree 5 vertices, and three degree 6 vertices.

Since $\Delta(T[EDG(P_n)]) = 6$ by the upper bound of $\varphi(G)$, $\varphi(T[EDG(P_n)]) \le 7$.

Case(i):

If $\varphi(T[EDG(P_n)]) = 7$ then $T[EDG(P_n)]$ must have seven vertices of degree 6, which is unfeasible, as $T[EDG(P_n)]$ has only three vertices of degree 6. Consequently, $\varphi(T[EDG(P_n)]) \ne 7$.

Case(ii):

If $\varphi(T[EDG(P_n)]) = 6$ then $T[EDG(P_n)]$ must have six vertices of degree 5, which is also unfeasible, as $T[EDG(P_n)]$ has only two vertices of degree 5. Consequently, $\varphi(T[EDG(P_n)]) \ne 6$.

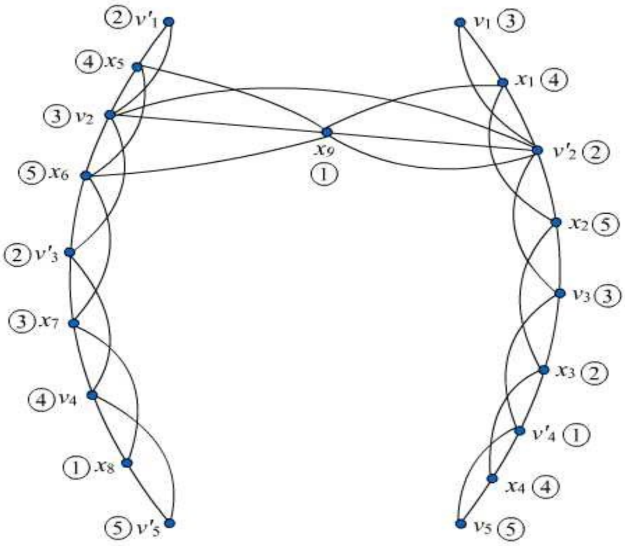

Figure 2 $\boldsymbol{T[EDG(P_5)]}$

Case(iii):

Due to adjacency of vertices in $T[EDG(P_n)]$ maximum of five b- vertices can be generated for the proper coloring. Consider the following 5-coloring (c_1, c_2, c_3, c_4, c_5) of $T[EDG(P_n)]$, assign the color c_1 to x_{2n-1}, assign the color c_2 to v_i' : $i = 1,2,3$. Assign the color c_3 to v_i : $i = 1,2,3$. Assign the color c_4 to x_1 *and* x_n, assign the color c_5 to x_2 *and* x_{n+1} and for the remaining vertices assign one of the allowed color - such color exists. The b - vertices $x_{2n-1}, v_2^{'}, v_2, x_1$ *and* x_{n+1} for the color classes c_1, c_2, c_3, c_4 *and* c_5 respectively produces the b - chromatic coloring.

Hence $\varphi(T[EDG(P_n)]) = 5, n \geq 4$

4. b-chromatic number of comb product of middle graph of an extension of duplicate graph of path with path graph P_n $(M[EDG(P_n)] \rhd P_n)$

Theorem 4.1

The b-chromatic number of comb product of middle graph of an extension of duplicate graph of path with path graph is, $\varphi(M[EDG(P_n)] \rhd P_n) = 5, n \geq 5$

Proof:

The comb product of middle graph of an extension of duplicate graph of P_n and the path graph has $4n^2$ -n vertices and $4n^2+n-1$ edges with maximum degree 7 and minimum degree 1. It contains 4n-1 vertices of degree 1, $4n^2-9n+6$ vertices of degree 2, 2n-6 vertices of degree 3 and 4, two vertices of degree 5 and 6, one vertex of degree 7.

Since $\Delta(M[EDG(P_n)] \rhd P_n) = 7$, by the upper bound of $\varphi(G)$, $\varphi(M[EDG(P_n)] \rhd P_n) \leq 8$.

Case(i):

If $\varphi(M[EDG(P_n)] \rhd P_n) = 8$ then $M[EDG(P_n)] \rhd P_n$ must have eight vertices of degree 7, which is unfeasible, as $M[EDG(P_n)] \rhd P_n$ has only one vertex of degree 7. Consequently, $\varphi(M[EDG(P_n)] \rhd P_n) \neq 8$.

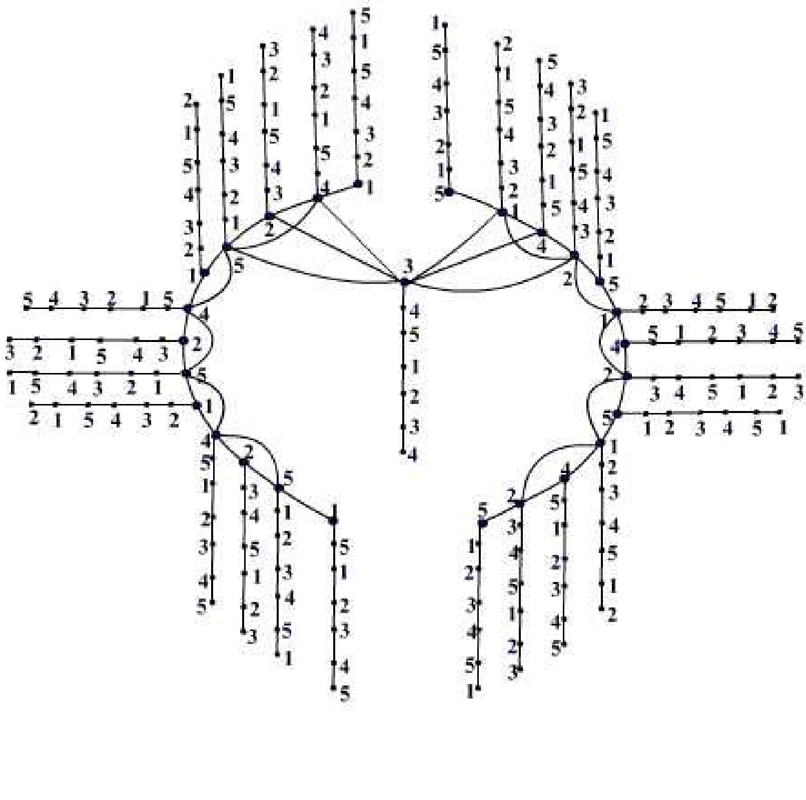

Figure 3 $M[EDG(P_7)] \rhd P_7]$

Case(ii):

If $\varphi(M[EDG(P_n)] \rhd P_n) = 7$ then $M[EDG(P_n)] \rhd P_n$ must have seven vertices of degree 6, which is unfeasible, as $M[EDG(P_n)] \rhd P_n$ has only two vertices of degree 6. Consequently, $\varphi(M[EDG(P_n)] \rhd P_n) \neq 7$.

Case(iii)

If $\varphi(M[EDG(P_n)] \rhd P_n) = 6$ then $M[EDG(P_n)] \rhd P_n$ must have six vertices of degree 5, which is unfeasible, as $M[EDG(P_n)] \rhd P_n$ has only two vertices of degree 5. Consequently, $\varphi(M[EDG(P_n)] \rhd P_n) \neq 6$.

Case(iv):

Due to adjacency of vertices in $M[EDG(P_n)] \rhd P_n$ maximum of five b- vertices can be generated for the proper coloring. Consider the following 5-coloring (k_1, k_2, k_3, k_4, k_5) of $M[EDG(P_n)] \rhd P_n$, assign the color c_1 to x_1 , x_3 , v'_1, assign the color c_2 to x_2 , v_2 , assign the color c_3 to x_{2n-1}, assign the color c_4 to v'_2, x_n assign the color c_5 to v_1 , x_{n+1} and the remaining vertices can be assigned from one of the allowed color - such color exists. The b - vertices x_1, x_2, x_{2n-1}, x_n and x_{n+1} for the color classes c_1, c_2, c_3, c_4 and c_5 respectively produces the b - chromatic coloring.

Hence $\varphi(M[EDG(P_n)] \rhd P_n) = 5, n \geq 5$

5. b-chromatic number of comb product of total graph of an extension of duplicate graph of path with path graph Pn ($T[EDG(P_n)] \rhd P_n$)

Theorem 3.1

The b-chromatic number of comb product of total graph of an extension of duplicate graph of path with path graph is, $\varphi(T[EDG(P_n)] \rhd P_n) = 6, n \geq 5$

Proof:

The comb product of middle graph of an extension of duplicate graph of path P_n and the path graph has $4n^2$ -n vertices and $4n^2+3n-3$ edges with maximum degree 7 and minimum degree 1. It contains 4n-1 vertices of degree 1, $4n^2-9n+2$ vertices of degree 2, four vertices of degree 3, two vertices of degree 4, 4n-12 vertices of degree 5, two vertices of degree 6 and three vertices of degree 7.

Since $\Delta(T[EDG(P_n)] \rhd P_n) = 7$, by the upper bound of $\varphi(G)$, $\varphi(T[EDG(P_n)] \rhd P_n) \leq 8$.

Case(i):

If $\varphi(M[EDG(P_n)] \rhd P_n) = 8$ then $M[EDG(P_n)] \rhd P_n$ must have eight vertices of degree 7, which is unfeasible, as $M[EDG(P_n)] \rhd P_n$ has only three vertices of

degree 7. Consequently, $\varphi(M[EDG(P_n)] \rhd P_n) \neq 8$.

Case(ii):

If $\varphi(M[EDG(P_n)] \rhd P_n) = 7$ then $M[EDG(P_n)] \rhd P_n$ must have seven vertices of degree 6, which is unfeasible, as $M[EDG(P_n)] \rhd P_n$ has only two vertices of degree 6. Consequently, $\varphi(M[EDG(P_n)] \rhd P_n) \neq 7$.

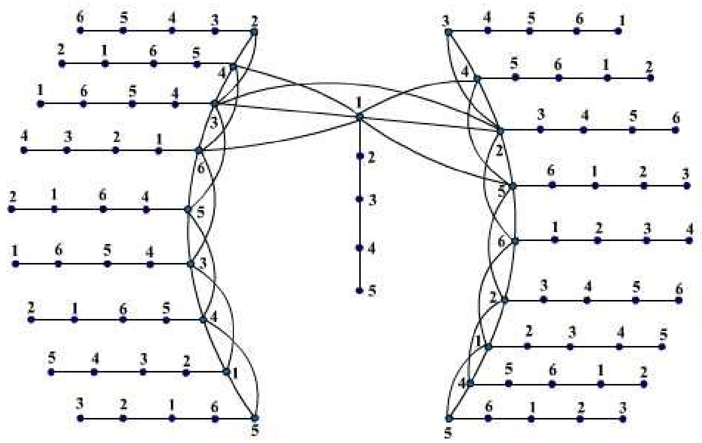

Figure 4 T$[EDG(P_5)] \rhd P_5]$

Due to adjacency of vertices in $T[EDG(P_n)] \rhd P_n$ maximum of five b- vertices can be generated for any proper coloring. Consider the following 5-coloring $(k_1, k_2, k_3, k_4, k_5)$ of $T[EDG(P_n)] \rhd P_n$, assign the color c_1 to x_{2n-1}, assign the color c_2 to $v_i' : i = 1,2,3$. Assign the color c_3 to $v_i : i = 1,2,3$. Assign the color c_4 to x_1 *and* x_n, assign the color c_5 to x_2 *and* x_{n+1} and for the remaining vertices assign one of the allowed color - such color exists. The b – vertices x_{2n-1}, v_2', v_2, x_1 *and* x_{n+1} for the color classes c_1, c_2, c_3, c_4 *and* c_5 respectively produces the b - chromatic coloring.

Hence $\varphi(T[EDG(P_n)] \rhd P_n) = 6, n \geq 5$

6.Conclusion

Applications of graph theory in computer science, electrical engineering, biology, and operations research are astounding. Scheduling, bandwidth allocation, and pattern matching are just a few of the useful uses for graph colouring. Here we have obtained the b – chromatic number of middle graph of an extension of duplicate graph of Path

(M[EDG(Pn)]), the total graph of an extension of duplicate graph of Path (T[EDG(Pn)]), the comb product of the middle graph of an extension of duplicate graph of Path graph with Pn ($M[EDG(P_n)] \rhd P_n$), and the comb product of the total graph of an extension of duplicate graph of Path graph with Pn ($T[EDG(P_n)] \rhd P_n$) are all determined in this paper.

7.References

1. Dafik, I.H. Agustin, D.A.R. Wardanai, E.Y. Kurniawat, R. Alfarisi, "On the Rainbow and Strong Rainbow Coloring of Comb Product Graphs", Acta Mechanica Slovaca 22 (3): 20 - 26, 2018
2. Effantin B, Kheddouci H, Discussion on the (partial) Grundy and b-chromatic numbers of graphs, Utilitas Mathematica, 80(2009), 65–89. On b-Chromatic Number of Some Line, Middle and Total Graph Families 125
3. Frank Harary, Graph Theory, Narosa Publishing Home, 2001.
4. Irving RW, Manlove DF, The b-chromatic number of a graph, Discrete Appl. Math., 91(1999), 127–141.
5. Michalak. D, 'On Middle And Total Graphs With Coarseness Number Equal I',Graph Theory, 1018,139-150,1981.
6. Sampathkumar E, On duplicate graphs, Journal of Indian Math. Soc., 37 (1973), 285-293.
7. Shobana Sarma C. and Thirusangu K., "Star Coloring of Extended Duplicate Graph of Path Graph Families", International Journal of Pure and Applied Mathematics 107(2016), 43-50
8. S. W. Saputro, N. Mardiana, and I. A. Purwasih, "The metric dimension of comb product graphs", Mat. Vesn., vol. 69 (4), pp. 248 258, 2017
9. Ulaganathan P P , K. Thirusangu and B. Selvam, Edge magic total labeling in extended duplicate graph of path, International Journal of Applied Engineering Research, 6(2011) 1211-1217.
10. Vernold V.J., Venkatachalam M., The b-chromatic number of corona graphs, Utilitas Mathematica, 88(2012), 299–307.