b –Coloring of the Comb product of Extended Duplicate graph of Cycle graph Networks with Path graph

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Abstract

The b – chromatic number which is $\varphi(G)$, the highest k such that graph can be b – colored using k colors. At least one vertex in every color group is adjacent to a vertex in every other color group. In this work we acquire b – chromatic number of the middle graph of an extension of duplicate graph of cycle graph (M[EDG(C_n)]), total graph of an extension of duplicate graph of cycle graph (T[EDG(C_n)]), Comb Product of middle graph of an extension of duplicate graph of cycle with (P_n) and Comb Product of total graph of an extension of duplicate graph of cycle with (P_n).

***Keywords* -** b-coloring, middle graph, extended duplicate graph, comb product, cycle, path.

1. Introduction

In this study, Graphs are finite, undirected, and have no loops or multiple edges, and color refers to vertex coloring.

The method of coloring a graph's vertices so that at least one vertex in each colour group has a neighbour in every other colour group is known as b-coloring. In G, the largest integer k that permits b-coloring is called a b-chromatic number. In [2,3,10], a number of intriguing coloring concepts and associated parameters are examined.

Manlove and Irving proposed the concept of b-coloring, and it has been shown that the problem of determining the b-chromatic number is NP-hard[4].

The upper bound of $\varphi(G)$ has also been established by Irving and Manlove,which is $\varphi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the highest degree of G.

A *duplicate graph of G = (V,E)* is denoted as DG = (V_1, E_1), the vertex set V_1 is the union of V and V' also $V \cap V' = \phi$. The function f from V to V' is bijective. The edge set E_1 of a duplicate graph has the edge u_1u_2' and $u_1'u_2$ if u_1u_2 is in E. The *extension of duplicate graph* of a duplicate graph, denoted by EDG, is defined as, any two vertices can be connected by an edge from V to any other vertex in V', except the end vertices. To make things easier, assume $u_2 \in V$ and $u_2' \in V'$ and thus the edge u_2u_2' is formed[6,7,9].

The *middle graph of G*, denoted by M(G) was introduced in 1981 and is defined as follows [5]. The vertex set of M(G) is V(G) ∪ E(G). Any two vertices *x*, *y* in M(G) are adjacent in M(G) if one of the following case holds.

(i) *x* and *y* are adjacent edges in G

(ii) *x* and *y* are incident in G.

The *total graph* of G, denoted by T(G), is defined as follows[5]. The vertex set of T(G) is V(G) ∪ E(G). Any two vertices *x*, *y* in the vertex set of T(G) are adjacent in T(G) if one of the following cases holds.

(i) *x*, *y* are in V(G) and *x* is adjacent to *y* in G.

(ii) *x*, *y* are in E(G) and *x*, *y* are adjacent in G.

(iii) *x* is in V(G), *y* is in E(G) and *x*, *y* are incident in G.

Let G and H represent two connected graphs. Let o be a vertex of H. The *comb product*[1,8] between graphs G and H, denoted by $G \rhd H$, is created by incorporating one instance of G along with |V(G)| instances of H. The i-th instance of H is connected to the i-th vertex of G at vertex o. The number of vertices in $G \rhd H$ is |V(G)|.|V(H)| and the number of edges is |V(G)|.|E(H)|+|E(G)|.

2. b-chromatic number of the middle graph of an extension of duplicate graph of cycle$(M[EDG(C_n)])$

A cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. A cycle with n vertices is denoted by C_n. The extended duplicate graph of cycle graph is denoted by EDG(C_n) with 2n vertices and 2n+1 edges.

Theorem 2.1

The b-chromatic number of middle graph of extended duplicate graph of cycle graph is $\varphi(M[EDG(C_n)]) = 5, n \geq 5$.

Proof:

Let the middle graph of extended duplicate graph of cycle C_n denoted by M[EDG(C_n)] with 4n+1 vertices and 6n+6 edges. Four vertices $x_1, x_2, x_{n+1}, x_{n+2}$ has degree 5, two vertices v_2 *and* v_2' have degree 3, v_i, v_i' $1 \leq i \leq n, i \neq 2$ has degree 2, one vertex x_{2n+1} have degree 6 and the remaining vertices has degree 4.

Since $\Delta(M[EDG(C_n)]) = 6$, $\varphi(M[EDG(C_n)]) \leq 7$.

Case(i)

Suppose $\varphi(M[EDG(P_n)]) = 7$ then $M[EDG(C_n)]$ must have seven vertices of degree 6, which is not possible because $M[EDG(C_n)]$ has only one vertex of degree 6. Hence, $\varphi(M[EDG(P_n)]) \neq 7$ and so $\varphi(M[EDG(C_n)]) \leq 6$.

Case(ii)

Suppose $\varphi(M[EDG(C_n)]) = 6$ then $M[EDG(C_n)]$ must have six vertices of degree 5, which is also not possible, as $M[EDG(C_n)]$ has only four vertices of degree 5. Hence, $\varphi(M[EDG(C_n)]) \neq 6$ and so $\varphi(M[EDG(C_n)]) \leq 5$.

Case(iii)

Clearly $M[EDG(C_n)]$ have 2n-4 vertices of degree 4. Due to adjacency of vertices in $M[EDG(C_n)]$ at most five b- vertices can be generated for any proper coloring. Consider the following 5-coloring (c_1, c_2, c_3, c_4, c_5) of $M[EDG(C_n)]$, assign the color c_1 to x_1 , v_2 . Assign the color c_2 to x_2 , v_1' *and* v_3'. Assign the color c_3 to x_{2n+1}, the color c_4 to v_1 , x_3, x_{n+1} . Assign the color c_5 to v_2' , x_{n+2} and for the remaining vertices assign one of the allowed color - such color exists. Then $x_1, x_2, x_{2n+1}, x_{n+1}, x_{n+2}$ are the b - vertices for the color classes c_1, c_2, c_3, c_4 *and* c_5 respectively.

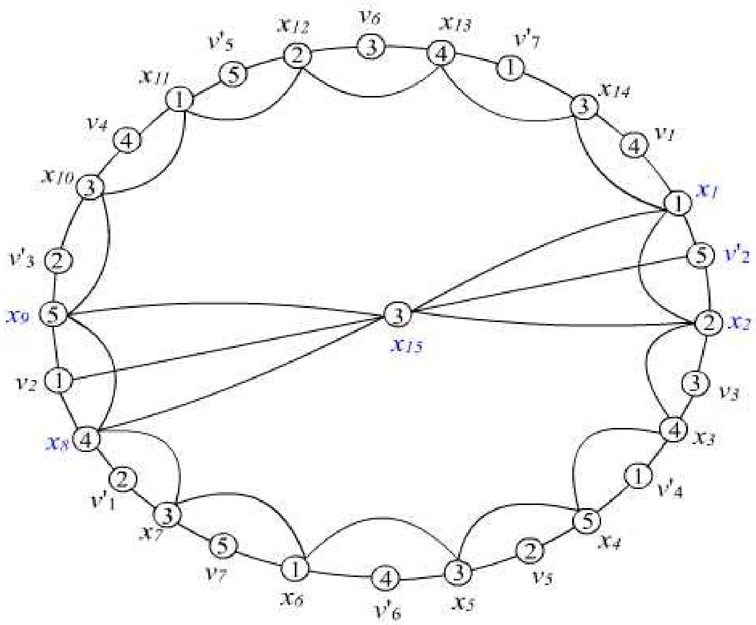

Figure 1 $M[EDG(C_7)]$

Therefore $\varphi(M[EDG(C_n)]) = 5, n \geq 5$

3. b-chromatic number of total graph of an extension of duplicate graph of cycle$(T[EDG(C_n)]$

Theorem 3.1

The b-chromatic number of total graph of extended duplicate graph of a cycle graph is, $\varphi(T[EDG(C_n)]) = 6, for\ n \geq 5$.

Proof:

T[EDG(C_n)] has 4n+1 vertices and 8n+7 edges with maximum degree 6 and minimum degree 4. It contains 4n-6 vertices of degree 4, four vertices of degree 5 and three vertices of degree 6.

Since $\Delta(T[EDG(C_n)]) = 6, \varphi(T[EDG(C_n)]) \leq 7$.

Case(i)

Suppose $\varphi(T[EDG(P_n)]) = 7$ then $T[EDG(C_n)]$ must have seven vertices of degree 6, which is impossible, since $T[EDG(C_n)]$ has only three verties of degree 6. Therefore, $\varphi(T[EDG(P_n)]) \neq 7$ and so $(T[EDG(C_n)]) \leq 6$

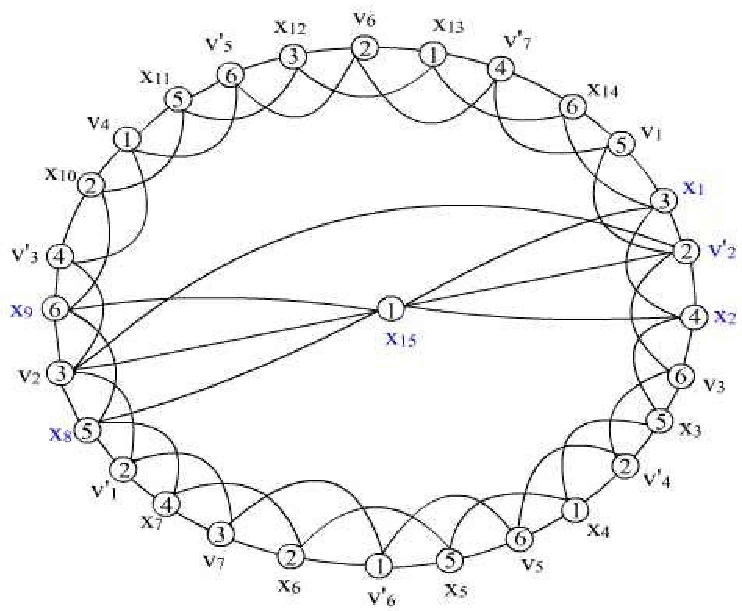

Figure 2 $\boldsymbol{T[EDG(P_7)]}$

Case(ii)

$T[EDG(C_n)]$ contains four vertices of degree 5 and three vertices of degree 6.Due to adjacency of vertices in $T[EDG(C_n)]$ at most six b- vertices can be generated for any proper coloring. Consider 6-coloring ($c_1, c_2, c_3, c_4, c_5, c_6$) of $T[EDG(C_n)]$.

Subcase(i) when n is odd

The color c_1 is assigned for x_{2n+1} the color c_2 is assigned for v_2', x_{n+3}, v_1', assign the color c_3 for x_1, v_2. Color c_4 is assigned for x_2, x_n, v_3'. The color c_5 is assigned for $x_{3\,,} v_1\, x_{n+1}$. Color c_6 is assigned for v_3 , x_{2n}, x_{n+2} and for the remaining vertices assign one of the allowed color - such color exists. Then $x_{2n+1},\ v_2', v_2,\ x_2, x_{n+1}, x_{n+2}$ are the b-vertices for the color classes c_1, c_2, c_3, c_4, c_5 *and* c_6 respectively.

Therefore $\varphi(T[EDG(C_n)]) = 6\,, n \geq 4$

Subcase(ii) when n is even

The color c_1 is assigned for x_{2n+1} the color c_2 is assigned for v_2', v_3', v_1', assign the color c_3 for x_1, v_2. Color c_4 is assigned for x_2, x_{2n}, x_{n+3}. The color c_5 is assigned for$x_{3\,,} v_1\, x_{n+1}$. Color c_6 is assigned for v_3 , $x_{n,} x_{n+2}$ and for the remaining vertices assign one of the allowed color - such color exists. Then $x_{2n+1},\ v_2', v_2,\ x_2, x_{n+1}, x_{n+2}$ are the b - vertices for the color classes c_1, c_2, c_3, c_4, c_5 *and* c_6 respectively.

Therefore $\varphi(T[EDG(C_n)]) = 6\,, n \geq 4$

4. b-chromatic number of comb product of middle graph of an extension of duplicate graph of cycle with path graph P$_n$ $(M[EDG(C_n)] \rhd P_n)$

Theorem 4.1

The b-chromatic number of comb product of middle graph of an extension of duplicate graph of cycle with path graph is, $\varphi(M[EDG(C_n)] \rhd P_n) \;=\; 6\,, n \;\geq\; 5$

Proof:

The comb product of middle graph of an extension of duplicate graph of C_n with the path graph has $4n^2+n$ vertices and $4n^2+3n+5$ edges with maximum degree 7 and minimum degree 2. It contains one vertex x_{2n+1} with degree 7, four vertices $x_1, x_2, x_{n+1}, x_{n+2}$ with degree 6. $x_i\ (1 \leq i \leq 2n)$ vertices with degree 5, two vertices v_2 *and* v_2' with degree 4, $v_i, v_i'\ (1 \leq i \leq n, i \neq 2)$ vertices with degree 3, and the remaining vertices with degree 2.

Since $\Delta(M[EDG(C_n)] \rhd P_n) = 7$, by the upper bound of $\varphi(G)$, $\varphi(M[EDG(C_n)] \rhd P_n) \leq 8$.

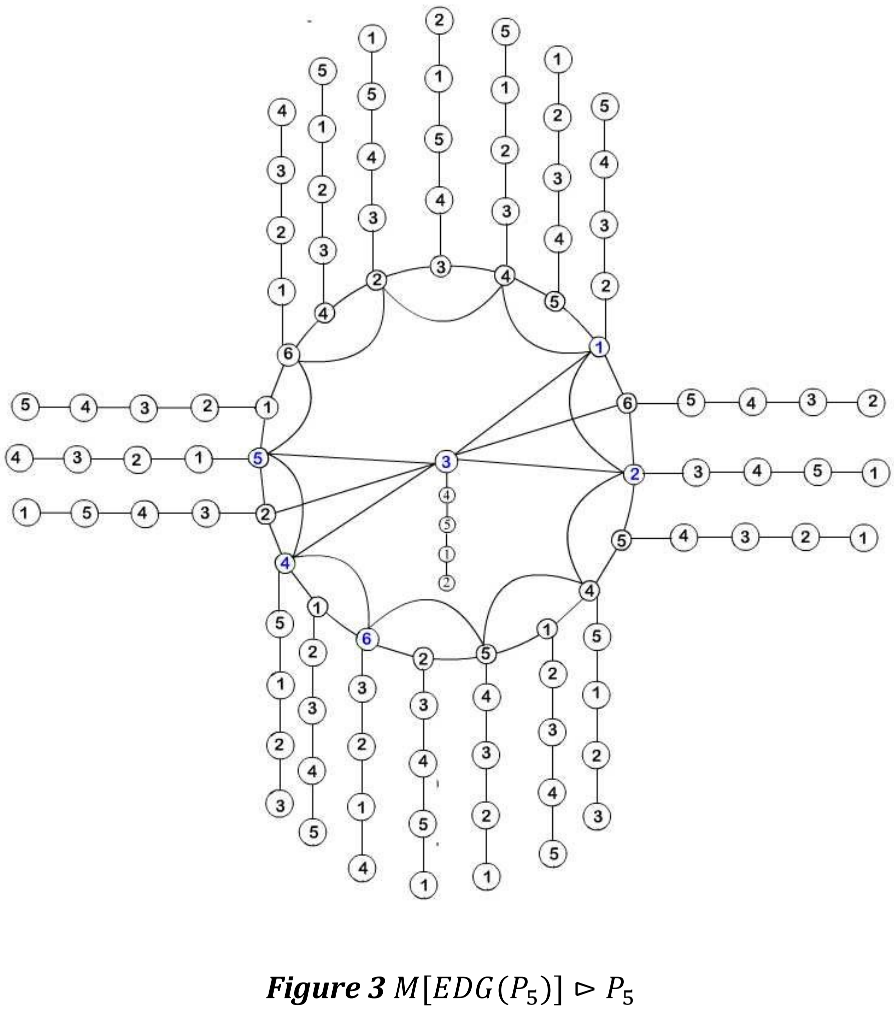

Figure 3 $M[EDG(P_5)] \rhd P_5$

Case(i):

If $\varphi(M[EDG(C_n)] \rhd P_n) = 8$ then $M[EDG(C_n)] \rhd P_n$must have eight vertices of degree 7, which is impossible since $M[EDG(C_n)] \rhd P_n$ has only one vertex of degree 7. Hence, $\varphi(M[EDG(C_n)] \rhd P_n) \neq 8$ and so $\varphi(M[EDG(C_n)] \rhd P_n) = 7$

Case(ii):

If $\varphi(M[EDG(C_n)] \rhd P_n) = 7$ then $M[EDG(C_n)] \rhd P_n$must have seven vertices of degree 6, which is also impossible since $M[EDG(C_n)] \rhd P_n$ has only two vertices of degree 6. Hence $\varphi(M[EDG(P_n)] \rhd P_n) \neq 7$ and so $\varphi(M[EDG(C_n)] \rhd P_n) = 6$

Case(iii)

$M[EDG(C_n)] \rhd P_n$ contains $x_i\,(1 \leq i \leq 2n)$ vertices of degree 5. Due to adjacency of vertices in $M[EDG(C_n)] \rhd P_n$at most six b- vertices can be generated for any proper coloring. Consider the following 6-coloring ($c_1, c_2, c_3, c_4, c_5, c_6$) of $M[EDG(C_n)] \rhd P_n$

Subcase(i) when n is odd

Assign the color c_1 to x_1 , $v_1' and\ v_3'$, assign the color c_2 to x_2 , $v_2\ and\ v_n$, assign the color c_3 to $x_{4n+1}\ and\ x_{2n+1}$, assign the color c_4 to $x_3, x_{2n}\ and\ x_{n+1}$, assign the color c_5 to $x_{n-1}, x_{n+2}, v_1\ and\ v_3$, assign the color c_6 to $x_n, x_{n+3}\ and\ v_2'$, and the remaining vertices can be assigned from one of the allowed color such that no adjacent vertices receive same color – such color exist. The b - vertices $x_1, x_2, x_{2n+1}, x_{n+1}, x_{n+2}\ and\ x_n$ for the color classes $c_1, c_2, c_3, c_4, c_5\ and\ c_6$ respectively produces the b - chromatic coloring.

Therefore$\varphi(M[EDG(C_n)] \rhd P_n) = 6\,, n \geq 5$

Subcase(ii) when n is even

Assign the color c_1 to x_1 , $v_1' and\ v_3'$, assign the color c_2 to x_2 , $v_2\ and\ v_n$, assign the color c_3 to $x_{2n-1}\ and\ x_{2n+1}$, assign the color c_4 to $x_{n+1}, v_1\ and\ v_3$, assign the color c_5 to $x_3, x_{6n+1}, x_n\ and\ x_{n+2}$, assign the color c_6 to $x_{2n}, x_{n+3}\ and\ v_2'$, and the remaining vertices can be assigned from one of the allowed color such that no adjacent vertices receive same color – such color exist. The b - vertices $x_1, x_2, x_{2n+1}, x_{n+1}, x_{n+2}\ and\ x_{2n}$ for the color classes $c_1, c_2, c_3, c_4, c_5\ and\ c_6$ respectively produces the b - chromatic coloring.

Therefore$\varphi(M[EDG(C_n)] \rhd P_n) = 6\,, n \geq 5$

5. b-chromatic number of comb product of total graph of an extension of duplicate graph of cycle with path graph $P_n(T[EDG(C_n)] \rhd P_n)$

Theorem 5.1

The b-chromatic number of comb product of total graph of an extension of duplicate graph of cycle with path graph is, $\varphi(T[EDG(C_n)] \rhd P_n) = 7$, $n \geq 5$.

Proof:

The comb product of total graph of an extended duplicate graph of C_n with the path graph has $4n^2+n$ vertices and $4n^2+5n+6$ edges with maximum degree 7 and minimum degree 2. It contains three vertices x_{2n+1} , v_2 *and* v_2' with degree 7, four vertices x_1 , x_2 , x_{n+1} , x_{n+2} with degree 6, x_i ($3 \leq i \leq 2n$, *for* $i \neq n+1$ *and* $n+2$), v_j , v_j' ($1 \leq j \leq n, j \neq 2$) vertices with degree 5, and the remaining vertices have degree 2.

Since $\Delta(T[EDG(C_n)] \rhd P_n) = 7$, by the upper bound of $\varphi(G)$, $\varphi(T[EDG(C_n)] \rhd P_n) \leq 8$.

Case(i):

If $\varphi(T[EDG(C_n)] \rhd P_n) = 8$ then $T[EDG(C_n)] \rhd P_n$ must have eight vertices of degree 7, which is impossible, as $T[EDG(P_n)] \rhd P_n$ has only three vertices of degree 7. Therefore, $\varphi(T[EDG(C_n)] \rhd P_n) \neq 8$ and so $\varphi(T[EDG(C_n)] \rhd P_n) = 7$

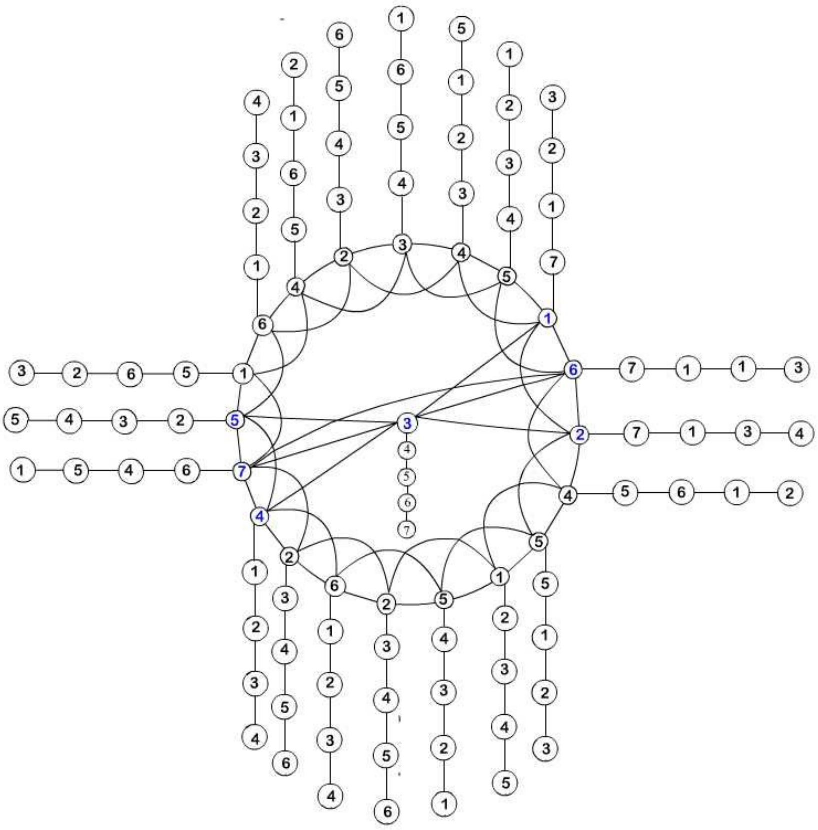

Figure 4 $\boldsymbol{T}[EDG(C_5)] \rhd P_5$

Case(ii)

Due to adjacency of vertices in $T[EDG(C_n)] \rhd P_n$ maximum of seven b- vertices can be generated for the proper coloring. Consider the following 7-coloring ($c_1, c_2, c_3, c_4, c_5, c_6, c_7$) of $T[EDG(C_n)] \rhd P_n$

Subcase(i) when n is odd

Assign the color c_1 to x_1 , x_{4n+2} *and* v_3', assign the color c_2 to x_2 , v_1' *and* x_{4n+4} , assign the color c_3 tox_{2n+1}, assign the color c_4 to v_3, x_{2n} *and* x_{n+1}, assign the color c_5 to x_3, x_{n+2} *and* v_1, assign the color c_6 to x_n, x_{n+3}, x_{4n+3} *and* v_2', assign the color c_7 to x_{2n+4}, x_{2n+3}, x_{2n+2} *and* $v_2$and the remaining vertices can be assigned from one of the allowed color such that no adjacent vertices receive same color – such color exist. The b - vertices $x_1, x_2, x_{2n+1}, x_{n+1}, x_{n+2}$, v_2' *and* v_2 for the color classes $c_1, c_2, c_3, c_4, c_5, c_6$ *and* c_7 respectively produces the b - chromatic coloring.

Therefore $\varphi(T[EDG(C_n)] \rhd P_n) = 7 , n \geq 5$

Subcase(ii) when n is even

Assign the color c_1 to x_1 , x_{2n} *and* v_3', assign the color c_2 to x_2 , v_1' *and* x_{n+3} , assign the color c_3 tox_{2n+1}, assign the color c_4 to v_2', x_{4n+3}, x_{4n+4} *and* x_{4n+5}, assign the color c_5 to x_3, x_{n+1} *and* v_1, assign the color c_6 to x_n, v_3, x_{n+2} *and* , assign the color c_7 to x_{4n+1}, x_{4n-1} *and* $v_2$and the remaining vertices can be assigned from one of the allowed color such that no adjacent vertices receive same color – such color exist. The b - vertices $x_1, x_2, x_{2n+1}, v_2', x_{n+1}, x_{n+2}$, *and* v_2 for the color classes $c_1, c_2, c_3, c_4, c_5, c_6$*and* c_7 respectively produces the b - chromatic coloring.

Therefore $\varphi(T[EDG(C_n)] \rhd P_n) = 7 , n \geq 5$

6.Conclusion

In this paper we have obtained the b – chromatic number of middle graph of an extension of duplicate graph of cycle (M[EDG(C_n)]), the total graph of an extension of duplicate graph of cycle (T[EDG(C_n)]), the comb product of the middle graph of an extension of duplicate graph of cycle graph with Pn($M[EDG(C_n)] \rhd P_n$), and the comb product of the total graph of an extension of duplicate graph of cycle graph with Pn($T[EDG(C_n)] \rhd P_n$).

7. References

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