MODAL OPERATORS ON BIPOLAR FERMATEAN FUZZY MATRICES

K. Lalitha[1], S. Sathya[2], S. Sarguna[3]

[1]Department of Mathematics, Annamalai University, Deputed to T.K.Government Arts College Virudhachalam, Tamil Nadu, India-606001, sudhan_17@yahoo.com

[2] Department of Mathematics, Annamalai University, ([2] Department of Mathematics, Panimalar Engineering College, Chennai)

Chidambaram, Tamil Nadu, India-608002., sathyasuresh10@gmail.com

[3]Department of Science and Humanities, Chennai Institute of Technology, Kundrathur, Chennai, Tamil Nadu, India-600069., sargunaphd21@gmail.com

Abstract

In this section, necessity and possibility operators are formulated for Bipolar Fermatean Fuzzy matrices, and various algebraic operations are discussed. Additionally, the complement of the matrix is studied using modal operator combinations.

Keywords: Bipolar Fermatean Fuzzy Set, Bipolar Fermatean Fuzzy Matrix, Modal Operators.

1.INTRODUCTION

The concept of fuzzy sets, introduced by Zadeh [14], has proven to be an essential tool for handling uncertainty and fuzziness. The concept of Pythagorean fuzzy sets and corresponding aggregation operations were first explored by Yager [10]. Atanassov [1] explored the concept of intuitionistic fuzzy sets and examined operations, modal operators, and topological aspects. The introduction of the modal operator in intuitionistic fuzzy matrices was made by Murugadas et al.[7] along with the development of aggregation operations. A set of properties concerning the bounded sum and bounded product of fuzzy matrices was proposed by Zhang et al.[13]. Meenakshi [5,6] explored the fundamental concepts of fuzzy matrices. Lalitha et al. [3,4] conducted a study that explored transitive closure and power convergence in bipolar intuitionistic fuzzy matrices. Silambarasan [8,9] explored some algebraic structures and investigated the commutative, associative, and distributive properties in Fermatean fuzzy matrices. The different types operators on intuitionistic fuzzy matrices was clarified by Boobalanet.al. [2] and relevant properties of IFMs have been studied. Muthuraji et al.[8] provided an explanation of how implication operators function on intuitionistic fuzzy matrices, and examined certain IFM properties. Their study addressed multiple aspects of bipolar fuzzy sets and supported the findings with numerical demonstrations. Senapati et al. [9] introduced Fermatean fuzzy sets.

2.PRELIMINARIES

In this section, let us recall some basic definitions ofFermatean fuzzy set and Bipolar Fermatean Fuzzy Set.

2.1 Definition

A fermatean fuzzy Set $(\mathcal{F}e\mathcal{F}\mathcal{S})$over the universal set is defined as $\mathcal{T} = \left\{t_p, t_{1_p}, t_{2_p} / t_{s_p} \in \mathcal{T}\right\}$where $t_{1_p}, t_{2_p}: \mathcal{T} \to [0,1]$. Also $0 \le \left(t_{1_p}\right)^3 + \left(t_{2_p}\right)^3 \le 1$.

2.2 Definition

A fermatean fuzzy matrix$(\mathcal{F}e\mathcal{F}\mathcal{M})$is defined as $\mathcal{T} = \langle t_{1_{pr}}, t_{2_{pr}} \rangle$ where $t_{1_{pr}}, t_{2_{pr}}: \mathcal{T} \to [0,1]$. Also

$0 \leq \left(t_{1_{pr}}\right)^3 + \left(t_{2_{pr}}\right)^3 \leq 1.$

2.3 Definition

A bipolar valued fuzzy set($\mathcal{BF}e\mathcal{S}$) of $\mathcal{T}$ is an object of the form $\mathcal{T} = \langle t_p, -t_{1_p}, t_{2_p}\rangle$.

Also, $t_{1_p}: \mathcal{T} \to [-1,0]$ and $t_{2_p}: \mathcal{T} \to [0,1]$. Where $-t_{1_p}, t_{2_p}$ denotes negative membership and positive membership respectively.

2.4 Definition

A Bipolar Fermatean Fuzzy Set($\mathcal{BF}e\mathcal{FS}$) is defined as $\mathcal{T} = \langle t_p, -t_{1_p}, t_{2_p}, -t_{3_p}, t_{4_p}\rangle$.
Where, $t_{1_p}, t_{3_p} \in [-1,0]$ and $t_{2_p}, t_{4_p} \in [0,1]$. Also $-1 \leq \left(t_{1_p}\right)^3 + \left(t_{3_p}\right)^3 \leq 0$, $0 \leq \left(t_{2_p}\right)^3 + \left(t_{4_p}\right)^3 \leq 1$

2.5 Definition

A BipolarFermatean Fuzzy Matrix($\mathcal{BF}e\mathcal{FM}$) is defined as $\mathcal{T} = \langle t_p, -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}}\rangle$. $\mathcal{BF}e\mathcal{FM}$ consists of element of $\mathcal{BF}e\mathcal{FS}$.

2.6 Definition

Let $\mathcal{T} = \langle t_p, -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}}\rangle$,$\mathcal{Q} = \langle q_p, -q_{1_{pr}}, q_{2_{pr}}, -q_{3_{pr}}, q_{4_{pr}}\rangle$be two $\mathcal{BF}e\mathcal{FM}s$ of same order. Then,

(a) If $\mathcal{T} \geq \mathcal{Q}$ then $t_{1_{pr}} \geq q_{1_{pr}}$,$t_{2_{pr}} \leq q_{2_{pr}}$, $t_{3_{pr}} \leq q_{3_{pr}}$,$t_{4_{pr}} \geq q_{4_{pr}}$.

(b) $\mathcal{T} \vee \mathcal{Q} = \begin{pmatrix} -max\left(t_{1_{pr}}, q_{1_{pr}}\right), min\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -min\left(t_{3_{pr}}, q_{3_{pr}}\right), max\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{pmatrix}$

(c) $\mathcal{T} \wedge \mathcal{Q} = \begin{pmatrix} -min\left(t_{1_{pr}}, q_{1_{pr}}\right), max\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -max\left(t_{3_{pr}}, q_{3_{pr}}\right), min\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{pmatrix}$

(d) $\mathcal{T} @ \mathcal{Q} = \begin{pmatrix} -\sqrt[3]{\frac{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3}{2}}, \sqrt[3]{\frac{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3}{2}}, \\ -\sqrt[3]{\frac{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3}{2}}, \sqrt[3]{\frac{\left(t_{4_{pr}}\right)^3 + \left(q_{4_{pr}}\right)^3}{2}} \end{pmatrix}$

(e) $\mathcal{T}^c = \langle -t_{3_{pr}}, t_{4_{pr}}, -t_{1_{pr}}, t_{2_{pr}}\rangle$

III. RESULTS OF BIPOLAR FERMATEAN FUZZY MATRIX USING MODAL OPERATORS

3.1 Definition

For any two bipolar fermatean fuzzy matrices $\mathcal{T}, \mathcal{Q} \in \mathcal{BF}e\mathcal{FM}$s of same order. Then,

(a) $\mathcal{T} \oplus \mathcal{Q} = \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -t_{3_{pr}} q_{3_{pr}}, \sqrt[3]{\left(t_{4_{pr}}\right)^3 + \left(q_{4_{pr}}\right)^3 - \left(t_{4_{pr}}\right)^3\left(q_{4_{pr}}\right)^3} \end{pmatrix}$

(b) $\mathcal{T} \otimes \mathcal{Q} = \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3\left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}} q_{4_{pr}} \end{pmatrix}$

(c) $\mathcal{T} \rightarrow_{\mathcal{BF}} \mathcal{Q} = \begin{pmatrix} -min\left(t_{3_{pr}}, q_{1_{pr}}\right), max\left(t_{4_{pr}}, q_{2_{pr}}\right), \\ -max\left(t_{1_{pr}}, q_{3_{pr}}\right), min\left(t_{2_{pr}}, q_{4_{pr}}\right) \end{pmatrix}$

(d) $\mathcal{T}^f = \left(\langle -\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f \rangle, \langle -\sqrt[3]{1-\left(1-\left(t_{3_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{4_{pr}}\right)^3\right)^f} \right) \rangle$

(e) $f\mathcal{T} = \left(-\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}, -\left(t_{3_{pr}}\right)^f, \left(t_{4_{pr}}\right)^f \right)$

3.2 Definition

For any bipolar fermatean fuzzy matrices $\mathcal{T} \in \mathcal{BFeFM}$s of same order. Then the necessity(□) and possibility (◇) operators are defined as follows,

(a) $\square\, \mathcal{T} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3} \right)$

(b) $\Diamond \mathcal{T} = \left(-\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}} \right)$

Proposition 3.1

If $\mathcal{T}$ and $\mathcal{Q}$ are two $\mathcal{BFeFM}$s of equal size, then the following equalities hold.

(a) $\square\,(\mathcal{T} \vee \mathcal{Q}) = \square\, \mathcal{T} \vee \square\, \mathcal{Q}$
(b) $\square\,(\mathcal{T} \wedge \mathcal{Q}) = \square\, \mathcal{T} \wedge \square\, \mathcal{Q}$
(c) $\Diamond(\mathcal{T} \vee \mathcal{Q}) = \Diamond\mathcal{T} \vee \Diamond\mathcal{Q}$
(d) $\Diamond(\mathcal{T} \wedge \mathcal{Q}) = \Diamond\mathcal{T} \wedge \Diamond\mathcal{Q}$

Proof:

(a) Let $\mathcal{T} = \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle$

$\mathcal{Q} = \langle -q_{1_{pr}}, q_{2_{pr}}, -q_{3_{pr}}, q_{4_{pr}} \rangle$ be two $\mathcal{BFeFM}s$ of same order.

$$\mathcal{T} \vee \mathcal{Q} = \begin{pmatrix} -max\left(t_{1_{pr}}, q_{1_{pr}}\right), min\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -min\left(t_{3_{pr}}, q_{3_{pr}}\right), max\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{pmatrix}$$

$$= \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle$$

$$\square\,(\mathcal{T} \vee \mathcal{Q}) = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3} \right) \quad (3.1)$$

$$\square\, \mathcal{T} \vee \square\, \mathcal{Q} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3} \right) \vee \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1-\left(q_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3} \right)$$

$$= \begin{pmatrix} -max\left(t_{1_{pr}}, q_{1_{pr}}\right), min\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -min\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{1_{pr}}\right)^3}\right), max\left(\sqrt[3]{1-\left(t_{2_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right) \end{pmatrix}$$

$$= \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3} \right) \quad (3.2)$$

From (3.1) and (3.2)

$\square\,(\mathcal{T} \vee \mathcal{Q}) = \square\, \mathcal{T} \vee \square\, \mathcal{Q}$

(c) $\Diamond(\mathcal{T} \vee \mathcal{Q}) = \left(-\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}} \right)$ (3.3)

$$\boxed{\Diamond}\mathcal{T} \vee \boxed{\Diamond}\mathcal{Q} = \left(-\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}}\right) \vee$$
$$\left(-\sqrt[3]{1-\left(q_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{4_{pr}}\right)^3}, -q_{3_{pr}}, q_{4_{pr}}\right)$$
$$= \left(\begin{array}{c} -max\left(\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{3_{pr}}\right)^3}\right), min\left(\sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{4_{pr}}\right)^3}\right), \\ -min\left(t_{3_{pr}}, q_{3_{pr}}\right), max\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{array}\right)$$
$$= \left(-\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}}\right) \qquad (3.4)$$

From (3.3) and (3.4)

$$\boxed{\Diamond}(\mathcal{T} \vee \mathcal{Q}) = \boxed{\Diamond}\mathcal{T} \vee \boxed{\Diamond}\mathcal{Q}$$

Similarly, we can prove (b) and (d).

Proposition 3.2

For any $\mathcal{BFeFMsT}$, the universal matrix $\mathcal{J}$ and the zero matrix $\mathcal{O}$. Then the equalities hold.

(a) $\boxed{\Diamond} \Box (\mathcal{T}) = \Box \boldsymbol{\mathcal{T}}$
(b) $\Box \boxed{\Diamond}(\mathcal{T}) = \boxed{\Diamond}\boldsymbol{\mathcal{T}}$
(c) $\Box (\boxed{\Diamond}\mathcal{J}) = \mathcal{J}$
(d) $\boxed{\Diamond}(\Box \mathcal{J}) = \mathcal{J}$
(e) $\Box (\boxed{\Diamond}\mathcal{O}) = \mathcal{O}$
(f) $\boxed{\Diamond}(\Diamond\mathcal{O}) = \mathcal{O}$
(g) $\Diamond\Diamond\mathcal{T} = \Diamond\mathcal{T}$
(h) $\boxed{\Diamond}\boxed{\Diamond}\mathcal{T} = \boxed{\Diamond}\mathcal{T}$
(i) $\Box \mathcal{O} = \mathcal{O}$
(j) $\boxed{\Diamond}\mathcal{O} = \mathcal{O}$
(k) $\Diamond\mathcal{J} = \mathcal{J}$
(l) $\boxed{\Diamond}\mathcal{J} = \mathcal{J}$

Proof:

(a) Let $\mathcal{T} = \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle$

$$\Box \mathcal{T} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-(t_2)^3}\right)$$
$$\boxed{\Diamond} \Box \mathcal{T} = \left(-\sqrt[3]{1-\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}\right)^3}, \sqrt[3]{1-\left(\sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)^3}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)$$
$$= \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right) = \Box \boldsymbol{\mathcal{T}}$$

Hence, $\boxed{\Diamond} \Box (\mathcal{T}) = \Box \boldsymbol{\mathcal{T}}$

Similarly, remaining can be prove analogically.

Proposition 3.3

For any $\mathcal{BFeFMsT}$.If $f > 0$ be any integer, then the following equalities holds.

(a) $\Box \mathcal{T}^f = (\Box \boldsymbol{\mathcal{T}})^f$

(b) $\Diamond\mathcal{T}^f = (\Diamond\boldsymbol{\mathcal{T}})^f$
(c) $\Box f\mathcal{T} = f \Box \mathcal{T}$
(d) $\Diamond f\mathcal{T} = f\Diamond\mathcal{T}$

Proof:

(a) $\mathcal{T}^f = \left(-\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f, -\sqrt[3]{1-\left(1-\left(t_{3_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{4_{pr}}\right)^3\right)^f}\right)$

$$\Box\mathcal{T}^f = \Box\left(-\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f, -\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}\right)$$

$$= \left(-\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f, -\sqrt[3]{1-\left(\left(t_{1_{pr}}\right)^f\right)^3}, \sqrt[3]{1-\left(\left(t_{2_{pr}}\right)^f\right)^3}\right) \quad (3.5)$$

$$(\Box\boldsymbol{\mathcal{T}})^f = \left(-\left(t_{1_{pr}}\right), \left(t_{2_{pr}}\right), -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)^{\boldsymbol{f}}$$

$$= \left(-\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f, -\sqrt[3]{1-\left(1-\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(\sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)^3\right)^f},\right)$$

$$= \left(-\left(t_{1_{pr}}\right)^f, \left(t_{2_{pr}}\right)^f, -\sqrt[3]{1-\left(\left(t_{1_{pr}}\right)^f\right)^3}, \sqrt[3]{1-\left(\left(t_{2_{pr}}\right)^f\right)^3}\right) \quad (3.6)$$

From (3.5) and (3.6)

$\Box\mathcal{T}^f = (\Box\boldsymbol{\mathcal{T}})^f$

(c) $\Box f\mathcal{T} = \left(-\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}, -\left(t_{3_{pr}}\right)^f, \left(t_{4_{pr}}\right)^f\right)$

$$= \begin{pmatrix} -\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}, \\ -\sqrt[3]{1-\left(\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(\left(t_{2_{pr}}\right)^3\right)^f} \end{pmatrix} \quad (3.7)$$

$$\Box\mathcal{T} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)$$

$$f\Box\mathcal{T} = \begin{pmatrix} -\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}, \\ -\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}\right)^{\boldsymbol{f}}\left(-\sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)^{\boldsymbol{f}} \end{pmatrix}$$

$$f \,\square\, \mathcal{T} = \begin{pmatrix} -\sqrt[3]{1-\left(1-\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(1-\left(t_{2_{pr}}\right)^3\right)^f}, \\ -\sqrt[3]{1-\left(\left(t_{1_{pr}}\right)^3\right)^f}, \sqrt[3]{1-\left(\left(t_{2_{pr}}\right)^3\right)^f} \end{pmatrix} \quad (3.8)$$

Similarly, we can prove (b) and (d).

Proposition 3.4

For anytwo$\mathcal{BF}$e$\mathcal{FM}$s $\mathcal{T}$and $\mathcal{Q}$. Then the following equalities holds.

(a) $(\mathcal{T} \wedge \mathcal{Q})^c = \mathcal{T}^c \vee \mathcal{Q}^c$
(b) $(\mathcal{T} \vee \mathcal{Q})^c = \mathcal{T}^c \wedge \mathcal{Q}^c$
(c) $(\square\, \mathcal{T}^c)^c = \boxed{\diamond}\mathcal{T}$
(d) $(\boxed{\diamond}\mathcal{T}^c)^c = \square\, \mathcal{T}$

Proof:

(a) $$(\mathcal{T} \wedge \mathcal{Q})^c = \begin{pmatrix} -min\left(t_{1_{pr}}, q_{1_{pr}}\right), max\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -max\left(t_{3_{pr}}, q_{3_{pr}}\right), min\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{pmatrix}^c = \langle -q_{1_{pr}}, q_{2_{pr}}, -q_{3_{pr}}, q_{4_{pr}} \rangle^c$$

$$= \langle -q_{3_{pr}}, q_{4_{pr}}, -q_{1_{pr}}, q_{2_{pr}} \rangle \quad (3.9)$$

$$\mathcal{T}^c \vee \mathcal{Q}^c = \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle^c \vee \langle -q_{1_{pr}}, q_{2_{pr}}, -q_{3_{pr}}, q_{4_{pr}} \rangle^c$$

$$= \langle -t_{3_{pr}}, t_{4_{pr}}, -t_{1_{pr}}, t_{2_{pr}} \rangle \vee \langle -q_{3_{pr}}, q_{4_{pr}}, -q_{1_{pr}}, q_{2_{pr}} \rangle$$

$$= \begin{pmatrix} -max\left(t_{3_{pr}}, q_{3_{pr}}\right), min\left(t_{4_{pr}}, q_{4_{pr}}\right), \\ -min\left(t_{1_{pr}}, q_{1_{pr}}\right), max\left(t_{2_{pr}}, q_{2_{pr}}\right) \end{pmatrix}$$

$$= \langle -q_{3_{pr}}, q_{4_{pr}}, -q_{1_{pr}}, q_{2_{pr}} \rangle \quad (3.10)$$

From (3.9) and (3.10)

$$(\mathcal{T} \wedge \mathcal{Q})^c = \mathcal{T}^c \vee \mathcal{Q}^c$$

(c) $$\square\, \mathcal{T}^c = \square\, \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle^c$$

$$= \square\, \langle -t_{3_{pr}}, t_{4_{pr}}, -t_{1_{pr}}, t_{2_{pr}} \rangle$$

$$= \left(-t_{3_{pr}}, t_{4_{pr}}, -\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3} \right)$$

$$(\square\, \mathcal{T}^c)^c = \left(-\sqrt[3]{1-\left(t_{3_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}} \right)$$

$$= \boxed{\diamond}\mathcal{T}$$

Proof of (b) and (d) are similar of (a) and (c).

Proposition 3.5

For any two $\mathcal{BF}$e$\mathcal{FM}$s$\mathcal{T}$and $\mathcal{Q}$. Then the following equalities holds.

(a) $\square\, \mathcal{T} \wedge \square\, \mathcal{Q} = \square\, (\mathcal{T}^c \vee \mathcal{Q}^c)^c$

(b) $\Box\,\mathcal{T} \vee \Box\,\mathcal{Q} = \Box\,(\mathcal{T}^c \wedge \mathcal{Q}^c)^c$
(c) $\diamond\mathcal{T} \wedge \diamond\mathcal{Q} = \diamond(\mathcal{T}^c \vee \mathcal{Q}^c)^c$
(d) $\diamond\mathcal{T} \vee \diamond\mathcal{Q} = \diamond(\mathcal{T}^c \wedge \mathcal{Q}^c)^c$

Proof:

(a) $$\Box\,\mathcal{T} \vee \Box\,\mathcal{Q} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right) \wedge \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1-\left(q_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right)$$

$$= \begin{pmatrix} -min\left(t_{1_{pr}}, q_{1_{pr}}\right), max\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -max\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{1_{pr}}\right)^3}\right), min\left(\sqrt[3]{1-(t_2)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right) \end{pmatrix}$$

$$= \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1-\left(q_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right) \quad (3.11)$$

$$(\mathcal{T}^c \vee \mathcal{Q}^c)^c = \langle -q_{3_{pr}}, q_{4_{pr}}, -q_{1_{pr}}, q_{2_{pr}} \rangle$$

$$\Box\,(\mathcal{T}^c \vee \mathcal{Q}^c)^c = \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1-\left(q_{2_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right) \quad (3.12)$$

From (3.11) and (3.12)

$\Box\,\mathcal{T} \wedge \Box\,\mathcal{Q} = \Box\,(\mathcal{T}^c \vee \mathcal{Q}^c)^c$

Similarly, we can prove other results.

Proposition 3.6

For any two $\mathcal{BF}e\mathcal{FM}s\mathcal{T}, \mathcal{Q}$ and$\mathcal{S}$. Then the following equalities holds.

(a) $\Box\left((\mathcal{T} \vee \mathcal{Q}) \wedge \mathcal{S}\right) = \left((\Box\,\mathcal{T} \vee \Box\,\mathcal{Q}) \wedge \Box\,\mathcal{S}\right)$
(b) $\diamond\left((\mathcal{T} \vee \mathcal{Q}) \wedge \mathcal{S}\right) = \left((\diamond\mathcal{T} \vee \diamond\mathcal{Q}) \wedge \diamond\mathcal{S}\right)$
(c) $\Box\left((\mathcal{T} \wedge \mathcal{Q}) \vee \mathcal{S}\right) = \left((\Box\,\mathcal{T} \wedge \Box\,\mathcal{Q}) \vee \Box\,\mathcal{S}\right)$
(d) $\diamond\left((\mathcal{T} \wedge \mathcal{Q}) \vee \mathcal{S}\right) = \left((\diamond\mathcal{T} \wedge \diamond\mathcal{Q}) \vee \diamond\mathcal{S}\right)$

Proof:

(a) $$\mathcal{T} \vee \mathcal{Q} = \begin{pmatrix} -max\left(t_{1_{pr}}, q_{1_{pr}}\right), min\left(t_{2_{pr}}, q_{2_{pr}}\right), \\ -min\left(t_{3_{pr}}, q_{3_{pr}}\right), max\left(t_{4_{pr}}, q_{4_{pr}}\right) \end{pmatrix}$$

$$(\mathcal{T} \vee \mathcal{Q}) \wedge \mathcal{S} = \langle -t_{1_{pr}}, t_{2_{pr}}, -t_{3_{pr}}, t_{4_{pr}} \rangle \wedge \langle -s_{1_{pr}}, s_{2_{pr}}, -s_{3_{pr}}, s_{4_{pr}} \rangle$$

$$= \begin{pmatrix} -min\left(t_{1_{pr}}, s_{1_{pr}}\right), max\left(t_{2_{pr}}, s_{2_{pr}}\right), \\ -max\left(t_{3_{pr}}, s_{3_{pr}}\right), min\left(t_{4_{pr}}, s_{4_{pr}}\right) \end{pmatrix}$$

$$\Box\left((\mathcal{T} \vee \mathcal{Q}) \wedge \mathcal{S}\right) = \left(-s_{1_{pr}}, s_{2_{pr}}, -\sqrt[3]{1-\left(s_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(s_{2_{pr}}\right)^3}\right) \quad (3.13)$$

From Proposition (3.1)

$$\Box\,\mathcal{T} \vee \Box\,\mathcal{Q} = \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)$$

$$\left((\square\, \mathcal{T} \vee \square\, \mathcal{Q}) \wedge \square\, \mathcal{S}\right)$$

$$= \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right)$$

$$\wedge \left(-s_{1_{pr}}, s_{2_{pr}}, -\sqrt[3]{1-\left(s_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(s_{2_{pr}}\right)^3}\right)$$

$$= \begin{pmatrix} -min\left(t_{1_{pr}}, s_{1_{pr}}\right), max\left(t_{2_{pr}}, s_{2_{pr}}\right), \\ -max\left(\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(s_{1_{pr}}\right)^3}\right), min\left(\sqrt[3]{1-\left(t_{2_{pr}}\right)^3}, \sqrt[3]{1-\left(s_{2_{pr}}\right)^3}\right) \end{pmatrix}$$

$$\left((\square\, \mathcal{T} \vee \square\, \mathcal{Q}) \wedge \square\, \mathcal{S}\right) = \left(-s_{1_{pr}}, s_{2_{pr}}, -\sqrt[3]{1-\left(s_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(s_{2_{pr}}\right)^3}\right) \quad (3.14)$$

From (3.13) and (3.14)

$$\square\left((\mathcal{T} \vee \mathcal{Q}) \wedge \mathcal{S}\right) = \left((\square\, \mathcal{T} \vee \square\, \mathcal{Q}) \wedge \square\, \mathcal{S}\right)$$

Similarly, we can prove other results.

Proposition 3.7

For any two $\mathcal{BF}e\mathcal{FM}s\mathcal{T}$ and $\mathcal{Q}$. Then the following equalities holds.

(a) $\square\,(\mathcal{T} \oplus \mathcal{Q}) = \square\, \mathcal{T} \oplus \square\, \mathcal{Q}$
(a) $\Diamond(\mathcal{T} \oplus \mathcal{Q}) = \Diamond\mathcal{T} \oplus \Diamond\Diamond\mathcal{Q}$
(b) $\square\,(\mathcal{T} \otimes \mathcal{Q}) = \square\, \mathcal{T} \otimes \square\, \mathcal{Q}$
(c) $\Diamond(\mathcal{T} \otimes \mathcal{Q}) = \Diamond\mathcal{T} \otimes \Diamond\mathcal{Q}$
(d) $\square\,(\mathcal{T} @ \mathcal{Q}) = \square\, \mathcal{T} @ \square\, \mathcal{Q}$
(e) $\Diamond(\mathcal{T} @ \mathcal{Q}) = \Diamond\mathcal{T} @ \Diamond\mathcal{Q}$

Proof:

$$\text{(a)}\quad \square\,(\mathcal{T} \oplus \mathcal{Q}) = \square \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -t_{3_{pr}} q_{3_{pr}}, \sqrt[3]{\left(t_{4_{pr}}\right)^3 + \left(q_{4_{pr}}\right)^3 - \left(t_{4_{pr}}\right)^3\left(q_{4_{pr}}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -\sqrt[3]{1-\left(\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}} q_{2_{pr}}\right)^3} \end{pmatrix}$$

$$\square\,(\mathcal{T} \oplus \mathcal{Q}) = \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3 - \left(q_{1_{pr}}\right)^3 + \left(t_{1_{pr}}\right)^3\left(q_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}} q_{2_{pr}}\right)^3} \end{pmatrix} \quad (3.15)$$

$$\square\, \mathcal{T} \oplus \square\, \mathcal{Q}$$

$$= \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1-\left(t_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(t_{2_{pr}}\right)^3}\right) \oplus \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1-\left(q_{1_{pr}}\right)^3}, \sqrt[3]{1-\left(q_{2_{pr}}\right)^3}\right)$$

$$= \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3 \left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -\sqrt[3]{1 - \left(t_{1_{pr}}\right)^3}\sqrt[3]{1 - \left(q_{1_{pr}}\right)^3}, \sqrt[3]{\left(\sqrt[3]{1 - \left(t_{2_{pr}}\right)^3}\right)^3 + \left(\sqrt[3]{1 - \left(q_{2_{pr}}\right)^3}\right)^3 - \left(\sqrt[3]{1 - \left(t_{2_{pr}}\right)^3}\right)^3 \left(\sqrt[3]{1 - \left(q_{2_{pr}}\right)^3}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt[3]{\left(t_{1_{pr}}\right)^3 + \left(q_{1_{pr}}\right)^3 - \left(t_{1_{pr}}\right)^3 \left(q_{1_{pr}}\right)^3}, t_{2_{pr}} q_{2_{pr}}, \\ -\sqrt[3]{1 - \left(t_{1_{pr}}\right)^3 - \left(q_{1_{pr}}\right)^3 + \left(t_{1_{pr}}\right)^3 \left(q_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{2_{pr}} q_{2_{pr}}\right)^3} \end{pmatrix} \quad (3.16)$$

From (3.15) and (3.16)

$\square\,(\mathcal{T} \oplus \mathcal{Q}) = \square\,\mathcal{T} \oplus \square\,\mathcal{Q}$

(c) $\square\,(\mathcal{T} \otimes \mathcal{Q}) = \square \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3 \left(q_{3_{pr}}\right)^3}, t_{4_{pr}} q_{4_{pr}} \end{pmatrix}$

$$= \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{1 - \left(t_{1_{pr}} q_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(\sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{1 - \left(t_{1_{pr}} q_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{2_{pr}}\right)^3 - \left(q_{2_{pr}}\right)^3 + \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3} \end{pmatrix} \quad (3.17)$$

$\square\,\mathcal{T} \otimes \square\,\mathcal{Q}$

$$= \left(-t_{1_{pr}}, t_{2_{pr}}, -\sqrt[3]{1 - \left(t_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{2_{pr}}\right)^3}\right) \otimes \left(-q_{1_{pr}}, q_{2_{pr}}, -\sqrt[3]{1 - \left(q_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(q_{2_{pr}}\right)^3}\right)$$

$$= \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{\left(\sqrt[3]{1 - \left(t_{1_{pr}}\right)^3}\right)^3 + \left(\sqrt[3]{1 - \left(q_{1_{pr}}\right)^3}\right)^3 - \left(\sqrt[3]{1 - \left(t_{1_{pr}}\right)^3}\right)^3 \left(\sqrt[3]{1 - \left(q_{1_{pr}}\right)^3}\right)^3}, \sqrt[3]{1 - \left(t_{2_{pr}}\right)^3}\sqrt[3]{1 - \left(q_{2_{pr}}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -t_{1_{pr}} q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3}, \\ -\sqrt[3]{1 - \left(t_{1_{pr}} q_{1_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{2_{pr}}\right)^3 - \left(q_{2_{pr}}\right)^3 + \left(t_{2_{pr}}\right)^3 \left(q_{2_{pr}}\right)^3} \end{pmatrix} \quad (3.18)$$

From (3.17) and (3.18)

$\square\,(\mathcal{T} \otimes \mathcal{Q}) = \square\,\mathcal{T} \otimes \square\,\mathcal{Q}$

Similarly, we can prove other results.

Proposition 3.8

For any two $\mathcal{BF}e\mathcal{FM}s\mathcal{T}$ and $\mathcal{Q}$. Then the following equalities holds.

(a) $\left(\square\,(\mathcal{T}^c \oplus \mathcal{Q}^c)\right)^c = \Diamond\mathcal{T} \otimes \Diamond\mathcal{Q}$

(b) $\left(\square\,(\mathcal{T}^c \otimes \mathcal{Q}^c)\right)^c = \Diamond\mathcal{T} \oplus \Diamond\mathcal{Q}$

(c) $\left(\Diamond(\mathcal{T}^c \oplus \mathcal{Q}^c)\right)^c = \Box\,\mathcal{T} \otimes \Box\,\mathcal{Q}$

(d) $\left(\Diamond(\mathcal{T}^c \otimes \mathcal{Q}^c)\right)^c = \Box\,\mathcal{T} \oplus \Box\,\mathcal{Q}$

Proof:

(a) $$\Box\,(\mathcal{T}^c \oplus \mathcal{Q}^c) = \Box\begin{pmatrix} -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}}, \\ -t_{1_{pr}}q_{1_{pr}}, \sqrt[3]{\left(t_{2_{pr}}\right)^3 + \left(q_{2_{pr}}\right)^3 - \left(t_{2_{pr}}\right)^3\left(q_{2_{pr}}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}}, \\ -\sqrt[3]{1 - \left(\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3 (q_3)^3}\right)^3}, \sqrt[3]{1 - \left(t_{4_{pr}}q_{4_{pr}}\right)^3} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}}, \\ -\sqrt[3]{1 - \left(t_{3_{pr}}\right)^3 - \left(q_{3_{pr}}\right)^3 + \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{4_{pr}}q_{4_{pr}}\right)^3} \end{pmatrix}$$

$$\left(\Box\,(\mathcal{T}^c \oplus \mathcal{Q}^c)\right)^c = \begin{pmatrix} -\sqrt[3]{1 - \left(t_{3_{pr}}\right)^3 - \left(q_{3_{pr}}\right)^3 + \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{4_{pr}}q_{4_{pr}}\right)^3}, \\ -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}} \end{pmatrix} \quad (3.19)$$

$$\Diamond\mathcal{T} \otimes \Diamond\mathcal{Q}$$
$$= \left(-\sqrt[3]{1 - \left(t_{3_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{4_{pr}}\right)^3}, -t_{3_{pr}}, t_{4_{pr}}\right) \otimes \left(-\sqrt[3]{1 - \left(q_{3_{pr}}\right)^3}, \sqrt[3]{1 - \left(q_{4_{pr}}\right)^3}, -q_{3_{pr}}, q_{4_{pr}}\right)$$

$$= \begin{pmatrix} -\sqrt[3]{1 - \left(t_{3_{pr}}\right)^3}\sqrt[3]{1 - \left(q_{3_{pr}}\right)^3}, \\ \sqrt[3]{\left(\sqrt[3]{1 - (t_4)^3}\right)^3 + \left(\sqrt[3]{1 - \left(q_{4_{pr}}\right)^3}\right)^3 - \left(\sqrt[3]{1 - \left(t_{4_{pr}}\right)^3}\right)^3\left(\sqrt[3]{1 - \left(q_{4_{pr}}\right)^3}\right)^3}, \\ -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}} \end{pmatrix}$$

$$\Diamond\mathcal{T} \otimes \Diamond\mathcal{Q} = \begin{pmatrix} -\sqrt[3]{1 - \left(t_{3_{pr}}\right)^3 - \left(q_{3_{pr}}\right)^3 + \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, \sqrt[3]{1 - \left(t_{4_{pr}}q_{4_{pr}}\right)^3}, \\ -\sqrt[3]{\left(t_{3_{pr}}\right)^3 + \left(q_{3_{pr}}\right)^3 - \left(t_{3_{pr}}\right)^3\left(q_{3_{pr}}\right)^3}, t_{4_{pr}}q_{4_{pr}} \end{pmatrix} (3.20)$$

From (3.19) and (3.20)

$\left(\Box\,(\mathcal{T}^c \oplus \mathcal{Q}^c)\right)^c = \Diamond\mathcal{T} \otimes \Diamond\mathcal{Q}$

Similarly, we can prove other results.

Conclusion

In this study, we explored the necessity and possibility operators on Bipolar Fermatean Fuzzy matrices and established their commutative, associative, and distributive properties

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