Spectral Bounds for the Signless Laplacian Matrix of Graphs: Laplacian Matrix, Upper bound along the i-th galactic Laplacian eigenvalue of a graph

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Abstract

The paper discusses the eigenvalues of a simple graph G with n vertices, focusing on its Laplacian eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \ldots\ldots\ldots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ and signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq \ldots\ldots\ldots \geq q_n(G) \geq 0$.

The main result establishes an upper conjugate for the i-th Laplacian eigenvalue:

$$\mu_i(G) \leq i-1+\underbrace{min}_{Ui} \max\{|N_H(v_k)N_H(v_j)|:v_kv_j \in E(H)\}$$

Here, $N_H(v_k)$ represents the set of neighbors of vertex v_k in the sub graph H, formed by removing an (i−1)-subset U_i of vertices from G. The paper also states that if the edge set E (H) is empty, then $\mu_i(G) \leq i-1$. Importantly, this bound is constrained to not exceed the order of the graph G.

Additionally, a significant inequality for a graph G's eigenvalues is proved in the study. In particular, it proves that: max $\{\mu_i(G),q_i(G)\} \leq dG(v_k) + 2d_G(v_i)$,
Where:

- $d_G(v_i)$ is the degree of vertex v_i in G,
- $d_G(v_k)$ is the degree of vertex v_k,
- $N=\{v_i, v_{i+1},\ldots,v_n\}$ is the subset of vertices starting from v_i to v_n,
- $\mu_i(G)$ and $q_i(G)$ are the i-th Laplacian and sign less Laplacian eigenvalues, respectively, of G.

This inequality provides an upper bound for the maximum of the i-th Laplacian and sign less Laplacian eigenvalues in terms of the vertex degrees, indicating a relationship between the eigenvalues and the structure of vertex degrees in the graph.

Moreover, we prove that max{μi(G), qi(G) } $\le \underbrace{max}_{i\le k\le n}$ {d G(v k)+ $\sum_{v_i v_j \in v_G(v_k)\cap N} \frac{d_G(v_j)}{d_G(v_k)}$}≤ 2dG(vi),

where d G (vi) is the i- th largest degree of G and N={vi, vi+1, ..., v n}.

Keywords:

Laplacian Matrix: L(G)=D−A, used for studying properties like connectivity and expansion.

- **Sign less Laplacian Matrix**: Q(G)=D+A, used for studying structural properties without considering signs.
- **Laplacian Spectrum**: The eigenvalues of L(G), related to the graph's connectivity and robustness.
- **sign less Laplacian Spectrum**: The eigenvalues of Q(G), used to understand other structural properties.
- **Diameter**: The maximum shortest path distance between any two vertices in the graph, indicating how far apart vertices are in the graph.

These concepts are fundamental in spectral graph theory and provide important insights into the structure and properties of a graph.

1. Introduction

Throughout this paper, let G = (V, E) be a simple undirected graph with vertex set

V (G) = {v_1, v_2... v_n} and edge set E (G). Also let $\acute{G}$ be the complement graph of G. For each

$v_i \in$ V (G), the set of neighbors of vertex v_i is denoted by N_G (v_i). Let d_G (v_i) be the degree of vertex v_i (i =1, 2 ….n).The diameter of G is the maximum distance between any two vertices of G. The Laplacian matrix of G is L (G) =D (G) – A (G), where D (G) is the diagonal matrix of vertex degrees of G and A(G) is the (0, 1)-adjacency matrix of G. It is well known that all the Laplacian eigenvalues of L (G) are non-negative.

Throughout this paper let μ_1 (G) ≥ μ_2 (G) ≥ …….≥ μ_{n-1} (G) ≥ μ_n (G) = 0 be the eigenvalues of L (G). The sign less Laplacian matrix of G is Q (G) =D (G) +A (G). It is easy to see that Q (G) is also positive semi definite and hence its eigenvalues can be arranged as

q_1 (G) ≥ q_2(G) ≥ ……….≥ q_n (G) ≥ 0.

A large body of research has been done on the Laplacian eigenvalues and how they relate to different graph features. Naturally, the extremal non-trivial eigenvalues have received the majority of attention in the study of Laplacian eigenvalues. In relation to the Laplacian eigenvalues of the underlying molecular graphs and the photoelectron spectra of saturated hydrocarbons (alkanes), both integer and real eigenvalues are used. Investigating the relationships between G's eigenvalues and graph theoretic characteristics is therefore important and required.

2. Preliminaries

Lemma 2.1: Let G be a simple graph with Laplacian spectrum $\{0 = \mu_n(G), \mu_{n-1}(G), ..., \mu_2(G), \mu_1(G)\}$. Then, the Laplacian spectrum of G is $\{0, n - \mu_1(G), n - \mu_2(G), ..., n - \mu_{n-2}(G), n - \mu_{n-1}(G)\}$.

Lemma 2.2: Let Abe a real symmetric matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and B be a principal sub matrix of A. If the eigenvalues of B are $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$, then $\alpha_i \geq \beta_i \geq \alpha_{n-m+i}$, for $i = 1, . . . , m$.

Lemma 2.3: For a symmetric and nonnegative matrix M, if $My \leq \beta y$, and then $\rho(M) \leq \beta$, where $\rho(M)$ is the ample eigenvalue of M, and that if $My < \beta y$, then $\rho(M) < \beta$.

- M is a symmetric, nonnegative matrix of size $n \times n$.
- y is a positive vector (meaning all its components are strictly greater than zero).
- β is a non-negative real number.
- The condition $My \leq \beta y$ implies that for each component of the vector My, we have $(My)i \leq \beta yi$, i.e., each component of the vector My is bounded by the corresponding component of βy.
- The goal is to show that under this condition, the largest eigenvalue $\rho(M)$ of M satisfies $\rho(M) \leq \beta$. Additionally, if the inequality is strict, i.e., $My < \beta y$, we want to prove that $\rho(M) < \beta$.

3. Upper bound along the i-th galactic Laplacian eigenvalue of a graph

Here by lead this bounds to the i-th larger Laplacian eigenvalue of a graph.

Theorem 3.1. Let G be a simple graph with more than indefinite edge, and let $\mu_i(G)$ be the i-th larger Laplacian eigenvalue of G, where $i \in \{1, 2, ..., n - 1\}$.

Then, $\mu_i(G) \leq i - 1 + \underbrace{min}_{Ui} \max \{|N_H(v_k) \cup N_H(v_j)| : v_k, v_j \in E(H)\}$ …………………… (1),

where $N_H(v_k)$ is the set of neighbour of vertex v_k in $V(H) = V(G) \backslash U_i$, U_i is any $(i - 1)$-subset of V (G).

Proof. Suppose that U_i is any $(i - 1)$-subset of V (G) and $V(H) = V(G) \backslash U_i$. Let $d_H(v_k)$ and $N_H(v_k)$, respectively, be the degree and the set of neighbour of vertex v_k in H. Then we have

$d_G(v_k) \leq d_H(v_k) + i - 1$ for all $v_k \in V(H)$. By Lemma2.2, we have

$\mu_i(G) \le \underbrace{min}_{Ui} \lambda_1(L_i)$,…….. (2) Where L_i is the main sub matrix of L (G) formed by deleting the rows and columns corresponding to the vertices in U_i, and $\lambda_1(L_i)$ is the largest eigenvalue of matrix L_i.

If $\lambda_1(L_i) \le i-1$, (1) already holds.

If $\lambda_1(L_i) > i-1$the following conditions follows.

Let x_s be the Eigen component of an eigenvector **x** corresponding to an eigenvalue $\lambda_1(L_i)$ of Li.

Also let $x_k = \underbrace{max}_{v_s \in V(H)} x_s$ (> 0) and $x_j = \underbrace{min}_{v_s \in N_H(v_k)} x_s$.

Let $W=N_H(v_k) \cap N_H(v_j)$, $U=N_H(v_k)\backslash W$and $V=N_H(vj)\backslash W$. From L_i **x**=$\lambda_1(L_i)$ **x** we get

$\lambda_1(L_i)x_k = d_G(v_k)x_k - \sum_{v_s \in N_H(v_k)} X_s = d_G(v_k)x_k - \sum_{v_s \in U} X_s - \sum_{v_s \in W} X_s$……………. (3)

$\lambda_1(L_i)x_j = d_G(v_k)x_j - \sum_{v_s \in N_H(v_j)} X_s = d_G(v_j)x_j - \sum_{v_s \in V} X_s - \sum_{v_s \in W} X_s$………….. (4)

For $x_k=x_j$, from (3) $\lambda_1(L_i) = d_G(v_k) - d_H(v_k) \le i-1$, a contradiction.

For $x_j > 0$, then $x_s > 0$ for any $v_s \in N_H(v_k)$. From (3), we get $\lambda_1(L_i) < d_G(v_k)$, a contradiction (by Lemma2.2, $\lambda_1(L_i) \ge d_G(v_k)$).

So we always have $x_k > 0$and $x_j \le 0$. From (3) and (4), we have

$\lambda_1(L_i)(x_k - x_j) = d_G(v_k)x_k - d_G(v_j)xj - \sum_{v_s \in U} X_s + \sum_{v_s \in W} X_s$

$\le d_G(v_k)x_k - d_G(v_j)x_j - |U|x_j + |V|x_k$

$= d_G(v_k)x_k - d_G(v_j)x_j - (d_H(v_k) - |W|)x_j + (d_H(vj) - |W|)x_k$

$\le (i - 1 + d_H(v_k) + d_H(v_j) - |W|)(x_k - x_j)$

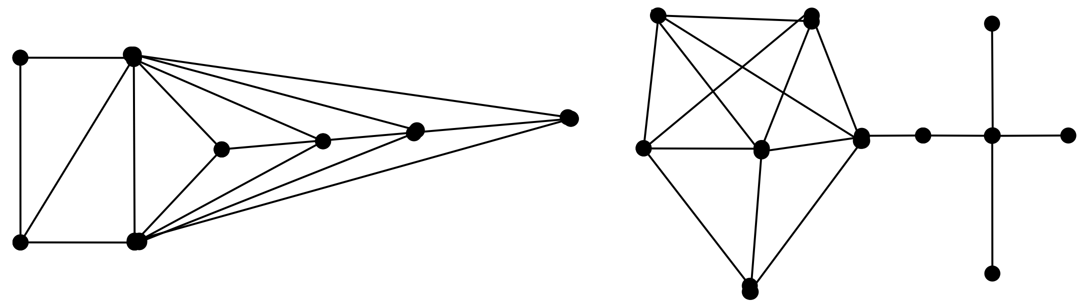
as $d_G(v_k) \le d_H(v_k) + i - 1$ and $d_G(v_j) \le d_H(v_j) + i - 1$

$= (i - 1 + |N_H(v_k) \cup N_H(v_j)|)(x_k - x_j)$.

Since $x_k - x_j > 0$, from the above, we get $\lambda_1(L_i) \le i - 1 + |N_H(v_k) \cup N_H(v_j)|$.

From (2) we get $\mu_i(G) \le \underbrace{min}_{Ui} \lambda_1(L_i) \le i-1+\underbrace{min}_{Ui} max \{ |N_H(v_k) \cup N_H(v_j)| : v_k v_j \in E(H) \}$.

This completes the proof.



G₁ G₂

Fig.1. Graphs G_1 and G_2.

Table1 Values of μ_2 (G) and the upper bound in (1).

	μ_2(G)	upper bound in (1).
G_1	6.26	7
G_2	6	6

The upper bound in (1) gives an integer value, which is always less than or equal to the order of G. Let $G_1$and G_2 be the graphs as shown in Fig. 1. Values of μ_2 (G) and the bound in (1) for G_1 and G_2 give Table 1.

Theorem 3.2. Let G be a simple graph with n vertices, and let v_p be an arbitrary vertex of G. Denote μ_2(G) as the second-largest eigenvalue of the Laplacian matrix of G. If the graph $G \setminus \{v_p\}$ (the graph obtained by removing vertex v_p) remains connected and $\mu_2(G)>k$ for some non-negative integer k, then the diameter of the graph $G \setminus \{v_p\}$ is at most n−k+3.

Proof. Suppose that there is an edge $v_r v_s \epsilon E(H)$ such that $1 + | N_H(v_r) \cup N_H(v_s) | =$
$\underbrace{min}_{p} \max \{1 + | N_H(v_i) \cup N_H(v_j) | : v_i v_j \epsilon E(H) \} \geq \mu_2(G) > k$, by Theorem 3.1,

Where $N_H(v_i)$ is the set of neighbors of vertex v_i in $H=G\setminus\{v_p\}$, v_p is any vertex of G. Thus, we have $|N_H(v_r) \cup N_H(v_s)| \geq k$. So, there are at most n –k vertices being adjacent to neither vertex v_r nor vertex v_s in $H=G\setminus\{vp\}$. Hence, the diameter of $G\setminus \{vp\}$ is at most n −k+3.

4. Merris' type upper bounds for μ_i(G) and q_i(G)

In this section, an upper bound to the i-th largest (sign less) Laplacian eigenvalue of graph G, which is an extension to Merris' upper bound for μ_1(G)

Theorem 4.1. For any graph G with non sporadic vertex,

$\mu_1(G) \leq q_1(G) \leq \underbrace{max}_{i \leq k \leq n} \{d_G(v_i) + \sum_{v_j \in N_G(v_i)} \frac{d_G(v_j)}{d_G(v_i)}\}$.

extend Theorem4.1to the following:

Theorem 4.2. Let G be a graph on n vertices with vertex degrees $d_G(v_1) \geq \cdots \geq d_G(v_n) \geq 1$, signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq \ldots \geq q_n(G)$ and Laplacian eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$. Then, $\max\{q_i(G), \mu_i(G)\} \leq \underbrace{max}_{i \leq k \leq n} \{d_G(v_i) + \sum_{v_j \in N_G(v_i) \cap N} \frac{d_G(v_j)}{d_G(v_k)}\}$,

where $N=\{v_i, v_{i+1}, ..., v_n\}$.

Proof. Prove the case of $\mu_i(G)$, since the case of $q_i(G)$ can be proved similarly.

Let $N=\{v_i, v_{i+1}, ..., v_n\}$, and also let L(G) be the Laplacian matrix of G such that the i-th row corresponds to the vertex v_i, where $1 \leq i \leq n$. Denote by B the Hermitian matrix of order $n - i + 1$ obtained from $D(G)^{-1}L(G)D(G)$ by deleting $i - 1$ rows and columns corresponding to the vertices $\{v_1, v_2, ..., v_{i-1}\}$. Let y be an $(n - i + 1)$-tuple vector, whose entries are 1, and let

$f = \underbrace{max}_{i \leq k \leq n} \{d_G(v_i) + \sum_{v_j \in N_G(v_i) \cap N} \frac{d_G(v_j)}{d_G(v_k)}\}$.

Then, $|B|\, y \leq f\, y$, where $|B|$ stands for the matrix whose entries are absolute values of the entries of B. Note that $D(G)^{-1}L(G)D(G)$ and L(G) share the same eigenvalues. By Lemmas2.2–2.3,

$\mu_i(G) \leq \rho(B) \leq \rho(|B|) \leq f = \underbrace{max}_{i \leq k \leq n} \{d_G(v_i) + \sum_{v_j \in N_G(v_i) \cap N} \frac{d_G(v_j)}{d_G(v_k)}\}$.

If $\mu_i(G) = f$, by Lemma2.3 we have $|B|\, y = f\, y$

, which implies that $d_G(v_i) + \sum_{v_j \in N_G(v_i) \cap N} \frac{d_G(v_j)}{d_G(v_i)} = \cdots = d_G(v_n) + \sum_{v_j \in N_G(v_n) \cap N} \frac{d_G(v_j)}{d_G(v_n)}$.

Corollary 4.3: If G is connected with the second largest maximum degree of G being equal to 2, then $q_2(G) < 4$ and $\mu_2(G) < 4$.

Proof. Let G be a connected graph with the second-largest maximum degree equal to 2. We are tasked with showing that $q_2(G) < 4$ and $\mu_2(G) < 4$, where $q_2(G)$ and $\mu_2(G)$ represent the second smallest nonzero eigenvalue of the Laplacian matrix and the second-largest Laplacian eigenvalue of G, respectively. By Theorem 4.2, which applies to graphs with the given degree condition, we know that for such graphs, both the second smallest nonzero eigenvalue $q_2(G)$ and the second-largest eigenvalue $\mu_2(G)$ are strictly less than 4. This follows directly from the fact that the second-largest maximum degree is bounded by 2 in this case, ensuring the eigenvalues satisfy the desired inequality.

Thus, we conclude that:

$q_2(G) < 4$and $\mu_2(G) < 4$.

Remark 4.4. It is well-established that:

$\mu_1(G) \leq q_1(G) \leq d_1+d_2$.

In the proof of **Theorem 4.2**, let B be the Hermitian matrix of order n−i+1, obtained from the Laplacian matrix L(G) by removing the i−1 rows and columns corresponding to the vertices

$\{v_1, v_2,\ldots,v_{i-1}\}$. By applying reasoning similar to that in **Theorem 4.2**, it can be shown that:

$\max\{q_i(G),\mu_i(G)\}\leq d_G(vi) + | N_G(vi)\cap\{v_{i,\ldots,\ldots}v_n\}|$,

where equality holds iff the following condition is satisfied for all vertices vi,vn:

$d_G(vi)+| N_G(v_i)\cap\{v_i,\ldots,v_n\}| =\ldots\ldots\ldots\ldots\ldots=d_G(v_n)+| N_G(v_n)\cap\{v_i,\ldots,v_n\}|$.

Using **Theorem 4.2** or the inequalities in (5), we can easily deduce that:

$q_i(G) \leq 2d_G(v_i)$ and $\mu_i(G) \leq 2d_G(v_i)$,

where the degrees of the vertices satisfy $d_G(v_1) \geq\ldots\ldots\ldots\ldots\geq d_G(v_n) \geq 1$.

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