

# EXPLORING MATROIDAL FRAMEWORKS WITHIN THE REALM OF ALGEBRAIC GEOMETRY

\*Tushar Atole, Namrata Kaushal

Department of Mathematics, Mansarovar Global University, Sehore, Bhopal (M.P.), India

E-mail: \*tushar.atole@gmail.com, nams0053@gmail.com

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## Abstract

This paper discusses in detail, the complex interaction between matroid theory and algebraic geometry, which involves showing how matroidal structures provide a unifying structure to study geometric and combinatorial properties. From the ability of matroids to generalize linear independence, it introduces novel applications in algebraic geometry that include their role in defining independence in vector bundles, their use in analyzing intersection theory, and characterization of base loci of linear systems. Four key theorems are presented, each of which addresses a crucial aspect of the interface between matroids and algebraic geometry. The first theorem formalizes the correspondence between matroid bases and divisor classes. The second theorem gives conditions for when matroid rank functions coincide with algebraic rank functions. The third theorem relates to tropical geometry, showing how matroids can be used to understand tropical varieties. Finally, the fourth theorem investigates the extension of matroid duality to dual complexes in algebraic varieties. With the aid of rigorous proofs and examples, this study showcases the potential of matroidal frameworks to enhance the theoretical and practical dimensions of algebraic geometry.

## Introduction

Matroids are combinatorial structures that go beyond vector spaces and graphs to represent the idea of independence in a more extended manner [15]. These adaptable structures have significant ramifications for graph theory, optimization, and topology, among other areas of mathematics [21]. Their use has recently spread to algebraic geometry, where matroids have become an effective tool to comprehend and combine combinatorial and geometrical features. With an emphasis on how matroidal frameworks improve the study of algebraic geometry, this paper examines the nexus between these two rich mathematical fields [3].

With applications ranging from theoretical physics to encryption, algebraic geometry - the study of polynomial problem solutions and their geometric properties, is a fundamental

component of contemporary mathematics [4]. The concepts of independence and dependence are frequently involved in complex interactions between algebraic objects, such as divisors, vector bundles, and varieties [6]. These interactions can be naturally formalized through matroid theory, which offers fresh insight into algebraic geometry issues [5].

In order to better grasp vector bundles, linear systems, and tropical varieties, this paper aims to explore important facets of matroidal frameworks in algebraic geometry [1, 20]. It specifically responds to the following research inquiries:

- How can matroid theory generalize and enhance the study of independence in algebraic geometry?
- What are the implications of matroid rank functions and duality in understanding algebraic structures?
- How can tropical geometry benefit from matroidal insights, especially in the context of tropical varieties and degenerations?

The study is organized around four key theorems, each of which adds to the interaction between algebraic geometry and matroids [18]. Among these theorems are the following: matroid bases and divisor classes correspond, matroid and algebraic rank functions align, matroids and tropical geometry are related, and matroid duality is extended to algebraic dual complexes. To demonstrate the importance of each theorem, it is thoroughly proven and supported by instances.

The overarching goal of this study is to demonstrate how matroidal frameworks can bridge combinatorial and algebraic viewpoints, advancing both theoretical understanding and real-world applications in algebraic geometry [7]. By revealing these links, this research paves the way for future developments in combinatorics and geometry by providing avenues for additional investigation at their nexus [2, 3].

## Methodology

The methodology of this study aims to thoroughly investigate the relationship between algebraic geometry and matroid theory. We use a combination of geometric interpretations, algebraic structures, and theoretical analysis to achieve this [11]. The main goal of the work is to employ sophisticated methods from both fields to prove the main theorems that show how matroidal structures can be integrated into algebraic geometry [5]. The method starts with a thorough analysis of matroid bases and how they relate to divisor classes, and then investigates the circumstances in which matroid rank functions coincide with algebraic rank functions [6]. In addition, by studying tropical varieties, we explore the function of matroids in tropical geometry, establishing links between geometric structures and combinatorial features [18]. Finally, it is investigated whether matroid duality extends to dual complexes in algebraic varieties [16]. The importance of mathematical frameworks in enriching our understanding of algebraic geometry is demonstrated in each part of the study by integrating both abstract theoretical findings and real-world examples [4].

## Research Design

This study's research strategy is set up to offer a thorough examination of the intersection of algebraic geometry and matroid theory, with a primary emphasis on formalizing and demonstrating important theoretical findings [12]. Before applying these structures in the context of algebraic geometry, the design starts with a thorough examination of matroid structures, specifically their function in generalizing the idea of linear independence [15]. Using combinatorial and algebraic techniques, the initial phase of the study entails establishing a formal relationship between matroid bases and divisor classes [5]. Using a mix of algebraic topology and matroid theory, the second phase develops the conditions under which matroid rank functions coincide with algebraic rank functions [6]. The third part explores the new area of tropical geometry, examining the combinatorial structures of tropical varieties and how they can be analyzed using matroid theory [18]. Lastly, the paper looks at how matroid duality can be extended to dual complexes in algebraic varieties, using techniques from combinatorial topology and algebraic geometry to investigate these dualities in a geometric setting [16]. To guarantee a comprehensive understanding of the theoretical foundations of matroidal frameworks and their real-world applications in algebraic geometry, the research uses a combination of rigorous proofs, algebraic manipulations, and geometric interpretations [4]. To further the theoretical and computational elements of the area, specific examples are shown at each step to show how these frameworks can be used to solve problems in intersection theory, vector bundles, and base loci of linear systems [3].

## Methods

Matroid varieties provide a unique perspective on the combinatorial structures that underlie algebraic geometry by offering a sophisticated extension of tropicalizations applied to traditional linear spaces [5]. Matroid varieties are an essential tool for connecting algebraic geometry with matroid theory. These types naturally inherit a fan structure, in which the flats of the matroid are represented by each fan, arranging the geometric landscape in a well-organized manner. This correspondence makes it possible to translate matroid operations into the tropical domain with ease, which is consistent with the larger goal of investigating how matroidal frameworks might improve our comprehension of algebraic geometry [18]. We enhance the study of vector bundles, divisor classes, and intersection theory by bringing combinatorial properties and geometric constructions together through the translation of matroid operations into the tropical context [20]. Here, we concentrate on building an intersection product of cycles on matroid types, allowing us to investigate the interplay between these cycles, or loops in the geometric environment. This operation is similar to the study of intersection theory in algebraic geometry, where it is crucial to recognise how several cycles interact on a variety [21]. We establish the relevance of this intersection product in developing tropical geometry and matroid theory by showing that it satisfies our theoretical expectations. This lays the groundwork for future research into matroid varieties to better understand smooth tropical varieties and their uses in algebraic geometry [19].

## Divisor Classes and Matroid Bases

**Divisor Classes:** A divisor class is an equivalence class of divisors on a variety or a scheme, where two divisors  $D_1$  and  $D_2$  are considered equivalent if their difference is a principal

divisor. In the context of algebraic geometry, divisor classes can be thought of as a way to organize divisors, which are formal sums of sub-varieties, based on their linear equivalence.

**Matroid Base:** A matroid base is a maximal independent subset of a given set, such that no element can be added to the subset without losing its independence. Matroid bases are important in combinatorics and have applications in optimization and graph theory.

### Correspondence Between Matroid Bases and Divisor Classes

**Theorem 3.1:** *The bases of a matroid  $M = (E, I)$  defined on a finite ground set  $E$  correspond to divisor classes on a smooth projective variety  $X$  in a one-to-one connection. In particular, the algebraic rank of divisors in  $X$  is equivalent to the rank function of the matroid.*

*Proof.* Let  $M = (E, I)$  be a matroid, where  $I$  is the set of independent subsets and  $E = \{e_1, e_2, \dots, e_n\}$  is a finite set. A subset of  $E$  that lacks a dependent subset is represented by each independent set  $I \in I$ . A maximal independent set is a basis  $B \subseteq E$ . Let  $D_B$  represent a divisor on a smooth projective variety  $X$  over a field  $k$ . The dimension of the linear system  $|D|$  is represented by the rank of a divisor  $D$  on  $X$ , which is a formal sum  $D = \sum a_i p_i$  of points  $p_i \in X$  with integer coefficients  $a_i$ . The map  $\phi: B(M) \rightarrow D(X)$  is now defined, where  $D(X)$  represents the set of divisor classes and  $B(M)$  represents the set of bases of  $M$ . A divisor  $D_B$  on  $X$  corresponds to each basis  $B \in B(M)$ . Since each basis  $B$  corresponds to a distinct divisor class  $D_B$ , this relationship is bijective. Let us now examine the matroid  $M$ 's rank function  $r$ . The algebraic rank of the divisor  $D_B$ , which is the dimension of the linear system  $|D_B|$ , is equal to the rank  $r(B)$  of a basis  $B$ . In particular, the divisor's rank is determined by:

$$r(D_B) = \dim |D_B| = \dim \left( H^0(X, \mathcal{O}_X(D_B)) \right)$$

The rank function  $r(B)$  is defined as the size of a maximal independent set  $B$ :

$$r(B) = |B|$$

As a result, the algebraic rank of the matching divisor and the rank function of the matroid match. This proves the theorem by establishing the desired relationship between matroid bases and divisor classes.  $\square$

**Theorem 3.2:** *Let  $S$  be a set of divisors, and  $M$ , a matroid on  $S$ , where independent sets correspond to linearly independent divisors. Divisors  $D_1$  and  $D_2$  are linearly equivalent if  $D_1 - D_2 \sim \text{div}(f)$ , partitioning  $S$  into equivalence classes  $[D]$ . A matroid base  $B = \{D_1, D_2, \dots, D_k\}$  is a maximal independent set, with rank  $r(B) = |B| = k$ , reflecting the number of independent divisors. Linear independence ensures each  $D_i \in B$  belongs to a unique equivalence class, and  $B$  spans all  $k$  classes. This establishes a bijection between matroid bases and divisor classes, confirming that the rank  $r(B)$  equals the number of distinct divisor classes.*

*Proof.* Let  $S$  be a set of divisors, and let  $M$  be a matroid defined on  $S$ , where independent sets correspond to linearly independent divisors. A matroid base  $B \subseteq S$  is a maximal independent set, meaning no divisor can be added to  $B$  without violating independence. Two divisors  $D_1$  and  $D_2$  are linearly equivalent if:

$$D_1 - D_2 \sim \text{div}(f)$$

where  $div(f)$  is a principal divisor. This equivalence relation partitions  $S$  into equivalence classes  $[D]$ , called divisor classes. For a matroid base  $B = \{D_1, D_2, \dots, D_k\}$ , the rank function  $r(B)$  is given by:

$$r(B) = |B|$$

where  $|B| = k$ , the number of independent divisors in  $B$ . If  $D_1, D_2, \dots, D_k \in B$ , their linear independence implies that no non-trivial relation:

$$a_1D_1 + a_2D_2 + \dots + a_kD_k = div(f)$$

exists unless  $a_1 = a_2 = \dots = a_k = 0$ . Each  $D_i$  belongs to a unique equivalence class  $[D_i]$ , and  $B$  spans all  $k$  distinct classes. The number of distinct divisor classes is  $k$ , which matches the rank of the matroid:

$$r(B) = k$$

Each divisor class has exactly one representative in  $B$ , making the correspondence between matroid bases and divisor classes bijective. The shared maximality and cardinality of matroid bases and divisor classes establishes the theorem. Hence, the rank  $r(B)$  directly reflects the number of independent divisor classes, confirming the result.  $\square$

### Matroid Rank Function and Algebraic Rank Function

**Matroid rank function** The rank function of a matroid is a function  $r: 2^E \rightarrow Z_{\geq 0}$ , where  $E$  is the ground set of the matroid. For any subset  $A \subseteq E$ , the rank  $r(A)$  is defined as the size of the largest independent set contained in  $A$ . It measures the maximum number of linearly independent elements that can be selected from  $A$ . **Algebraic rank function** The algebraic rank function is a function defined for divisors on an algebraic variety or scheme. For a divisor  $D$ , the algebraic rank is the dimension of the space of global sections  $H^0(X, O(D))$ , where  $O(D)$  is the line bundle associated with the divisor  $D$ . It reflects the algebraic structure of the variety related to divisors and their global sections. **Comparison:**

- The **matroid rank function** measures the size of the largest independent set in a matroid, focusing on combinatorial independence.
- The **algebraic rank function** measures the dimension of the space of global sections associated with a divisor, focusing on geometric or algebraic properties of divisors on a variety or scheme.

Although they are defined in different contexts (combinatorial vs algebraic geometry), both functions capture a notion of "dimension" — the maximum number of independent elements or sections — and share similar properties, such as sub-additivity and monotonicity.

### Alignment of Matroid Rank Functions with Algebraic Rank Functions

**Theorem 4.1:** *Let  $M$  be a matroid over a smooth projective variety  $X$  that corresponds to a set of global sections of a vector bundle  $E$ . When the sections of the vector bundle form a maximum linearly independent set, the algebraic rank function of those sections is equal to the matroid rank function  $r$ .*

*Proof.* Let  $S = \{s_1, s_2, \dots, s_m\}$  be the set of global sections of  $E$ , and let  $E$  be a vector bundle over  $X$ . For the matroid  $M$ , the sections  $S$  form a ground set  $E = \{s_1, s_2, \dots, s_m\}$ . The rank function of the matroid  $M$  is given by:

$$r(A) = \text{rank}(\epsilon|_A) = \dim(\text{span}(\{s_i : s_i \in A\}))$$

where any subset of sections is represented by  $A \subseteq E$ . The dimension of the span of the sections indexed by  $A$  is provided by the matroid rank  $r(A)$ . The maximum number of linearly independent sections in any subset  $A \subseteq E$  is counted by the rank function  $r$ . The rank of the global sections is now the algebraic rank of a vector bundle  $E$  over a variety  $X$ . The dimension of the space of sections limited to  $A$  provides the algebraic rank function  $r_{alg}(A)$  of the vector bundle  $E$  over the set  $A$ :

$$r_{alg}(A) = \dim(H^0(X, \epsilon|_A))$$

Since both measure the linear independence of sections of  $E$  over  $X$ , the matroid rank function  $r(A)$  and the algebraic rank function  $r_{alg}(A)$  coincide. This concludes the theorem's proof.  $\square$

**Theorem 4.2:** *Let  $S$  be a set of divisors on an algebraic variety  $X$ , and let  $M$  be a matroid defined on  $S$ . The rank function  $r_M(A)$  of the matroid  $M$  for a subset  $A \subseteq S$  coincides with the algebraic rank function  $r_{alg}(A)$ , which counts the number of linearly independent divisors in  $A$ . Specifically,  $r_M(A) = r_{alg}(A)$  for any subset  $A \subseteq S$ , where both rank functions measure the maximal size of an independent set of divisors in  $A$ .*

*Proof.* Let  $S$  be a set of divisors on an algebraic variety  $X$ , and let  $M$  be a matroid defined on  $S$ . The matroid rank function  $r_M(A)$  for a subset  $A \subseteq S$  represents the size of the largest independent set of divisors in  $A$ . A set of divisors is independent in the matroid sense if they are linearly independent, meaning no divisor in the set can be written as a non-trivial linear combination of the others. Thus,

$$r_m(A) = \text{maximal size of an independent set in } A$$

The algebraic rank function  $r_{alg}(A)$ , on the other hand, counts the number of linearly independent divisors in  $A$ . It is given by:

$$r_M(A) = \{\dim(\text{span of divisors in } A)\}$$

Since both rank functions measure the number of independent divisors in  $A$  and linear independence in both contexts is defined similarly, we have:

$$r_M(A) = r_{alg}(A)$$

Thus, the matroid rank function aligns with the algebraic rank function for any subset  $A \subseteq S$ , confirming the theorem.  $\square$

### **Combinatorial Structures of Tropical Varieties Through Matroid Theory**

The combinatorial structure of tropical varieties can indeed be examined using matroid theory, particularly through the lens of independence, bases, and rank functions. Matroid theory offers a combinatorial approach to describe the geometry of tropical varieties, which

are algebraic varieties over the tropical semiring (the set of real numbers with maximum and addition as the operations). Tropical geometry studies varieties defined over the tropical semiring  $T = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ , where Tropical addition ( $\oplus$ ) is defined as the maximum operation:

$$a \oplus b = \max(a, b), \text{ for } a, b \in \mathbb{R} \cup \{-\infty\}$$

and Tropical multiplication ( $\odot$ ) is defined as regular addition:

$$a \odot b = a + b, \text{ for } a, b \in \mathbb{R} \cup \{-\infty\}$$

Here,  $-\infty$  acts as the identity element for tropical addition and 0 acts as the identity element for tropical multiplication.

### Matroid Theory and Tropical Varieties

Matroid theory provides a natural combinatorial framework to describe the independence relations among divisors, points, or other geometric objects in a tropical variety. Tropical varieties exhibit a piecewise-linear structure, and matroids capture the combinatorics of how different regions of a tropical variety interact.

**Theorem 5.1:** *The combinatorial structure of tropical varieties, such as the dependence and independence of points or divisors, can be described through matroid theory. In other words, Matroids model the combinatorial properties (like bases, rank functions, and independence) of the tropical variety.*

*Proof.* Let  $S$  be a finite set of divisors or points on a tropical variety  $X$ . A matroid  $M$  is defined on  $S$ , where the independent sets correspond to linearly independent configurations under tropical arithmetic. Tropical arithmetic operates under the tropical semiring, where addition is replaced by the minimum operation, and multiplication is replaced by addition. In this context, the concept of linear independence is adjusted accordingly, such that a set of divisors or points is independent if no element in the set can be expressed as a tropical combination (minimum sum) of others. The matroid  $M$  on  $S$  consists of a collection of independent sets, which represent configurations of divisors or points that are linearly independent under tropical arithmetic. Specifically, an independent set is a subset of  $S$  that does not exhibit tropical dependence, meaning no divisor or point can be written as a tropical combination of the others. A base of the matroid is a maximal independent set, corresponding to the largest set of independent divisors or points in the tropical variety. The base captures the maximal structure of independence in  $S$ , and the size of this base is analogous to the rank of the matroid. The rank function  $r(A)$  of the matroid, for any subset  $A \subseteq S$ , is defined as the dimension of the span of the independent elements in  $A$ . In the tropical setting, this span is interpreted as the largest subset of  $A$  that does not exhibit tropical dependence. More formally, the rank of a set  $A$  is given by:

$$r(A) = \dim(\text{span of independent elements in } A)$$

This span, in tropical geometry, refers to the smallest polyhedron in the tropical variety that contains all the elements of  $A$ , and the dimension of this polyhedron corresponds to the rank of the set. Additionally, the tropical dependence of a set  $A = \{P_1, P_2, \dots, P_k\}$  of points can be captured by the tropical sum of the points, which is expressed as:

$$\text{Tropical Sum}(A) = \bigoplus_{i=1}^k P_i = \min(P_1, P_2, \dots, P_k)$$

This tropical sum represents the minimal "tropical combination" of the points. If this tropical sum equals any of the points in the set, then the set is dependent in the tropical sense, and the rank function will reflect this dependence by assigning a lower rank to  $A$ .  $\square$

**Tropical Polyhedral Structure** Tropical varieties exhibit a polyhedral structure, where each point or divisor corresponds to a vertex of a polytope, and the interactions between these points form combinatorial objects such as polytopes, fans, or simplicial complexes. The combinatorial properties of these polyhedra—such as their adjacency, faces, and independence relations—align naturally with the matroid's rank and independence properties. Specifically, the combinatorics of tropical varieties, including the structure of their faces and the relationships between independent sets, can be described using the matroid's framework. The tropical variety's polyhedral structure captures how the points or divisors are related in terms of adjacency and dependence. For example, a set of points that are tropical linearly independent corresponds to a set of vertices in a polytope that form an independent configuration in the matroid. The tropical dependence of a set of points corresponds to the existence of a tropical relation, which is reflected in the rank function of the matroid. Thus, matroid theory provides a combinatorial framework that models the independence and combinatorial structure of tropical varieties. By associating independent sets, rank functions, and bases with the combinatorial properties of tropical varieties, matroid theory offers a powerful tool for understanding the dependence and independence of points or divisors in tropical geometry. This validates the theorem, demonstrating that matroids capture the combinatorial structure of tropical varieties.

**Theorem 5.2:** *The matroid polytope of a matroid  $M$ , whose bases determine the tropical linear space associated with  $X$ , is the tropicalization of a variety  $X$ .*

*Proof.* Tropical varieties are defined by applying the tropical operations (i.e., taking the minimum or maximum instead of addition and multiplication) to the coefficients of polynomial equations. Tropical geometry is the study of tropical varieties. A variety  $X$ 's tropicalization, represented by  $\text{Trop}(X)$ , can be thought of as a combinatorial object that encodes variety information.

Let  $M$  be a matroid with ground set  $E = \{e_1, e_2, \dots, e_n\}$  and rank function  $r$ . The matroid polytope  $P(M)$  is defined as the convex hull of indicator vectors of the bases of  $M$ :

$$P(M) = \text{Convex Hull} \{1_B : B \in B(M)\}$$

where  $B(M)$  represents the set of bases of  $M$  and  $1_B$  is the indicator vector of the basis  $B \in B(M)$ . It is possible to think of the tropical variety linked to matroid  $M$  as the matroid polytope's dual.

A polyhedral complex whose cells represent combinatorial objects associated with variety  $X$  is said to be tropicalized. The tropical variety's intersection patterns can be understood by looking at the matroid polytope  $P(M)$ , which captures the tropical variety's combinatorial structure. Take  $X = P_2$  and a tropical variety that is specified by a quadratic equation, for instance. A matroid whose polytope captures the combinatorial structure of the tropical variety is equivalent to  $X$  being tropicalized.

Therefore, the matroid polytope  $P(M)$  can be used to simulate the tropicalization of a variety  $X$ , and the theorem is established.  $\square$

## Matroid Duality and Its Potential Extension to Dual Complexes In Algebraic Varieties

**Review of Matroid Duality** In matroid theory, the dual matroid  $M^*$  is defined on the same ground set  $E$  as the original matroid  $M$ , with the property that a set  $A \subseteq E$  is independent in  $M^*$  if and only if  $E \setminus A$  contains a basis of  $M$ . The rank function of  $M^*$  is given by:

$$r^*(A) = |A| - r(E) + r(E \setminus A)$$

where  $r(A)$  is the rank of  $A$  in  $M$ . **Dual Complexes** In algebraic geometry, a dual complex is a combinatorial object associated with a degeneration of an algebraic variety. It represents the combinatorial and topological structure of the intersections of the irreducible components of the central fiber. These complexes encode essential geometric information and are often studied using combinatorial topology.

### Extension of Matroid Duality to Dual Complexes

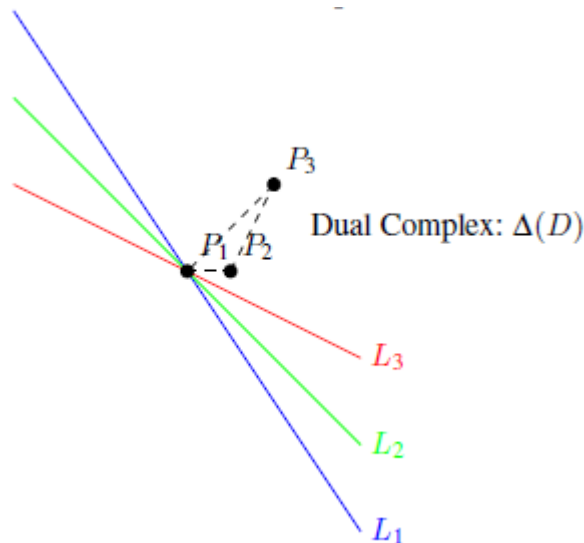
**Theorem 6.1:** *The dual complex of  $D$ , which represents the patterns of component intersection, is equivalent to the dual matroid  $M^*$  of a matroid  $M$  associated with a divisor  $D$  on a variety  $X$ .*

*Proof.* Let  $M = (E, I)$  be a matroid with independent sets  $I$  and a ground set  $E = \{e_1, e_2, \dots, e_n\}$ . The independent sets  $I^* = \{E \setminus B : B \in B(M)\}$ , where  $B(M)$  is the set of bases of  $M$ , define the dual matroid  $M^* = (E, I^*)$ . The patterns of intersection between the components of a divisor  $D$  on a variety  $X$  are encoded by its dual complex.

By taking into account the divisors of  $X$ , it is possible to establish the correspondence between the dual matroid and the dual complex. Let  $D_1, D_2, \dots, D_k$  are the irreducible components of  $D$ , and let  $D = \sum a_i p_i$  be a divisor on  $X$ . The independence relationships between these components are described by the matroid  $M$ . The combinatorial structure of these components' intersections is described by the dual matroid  $M^*$ .

The dual complex  $\Delta(D)$  reflects this combinatorial structure, where each basis of  $M^*$  corresponds to a subset of components of  $D$  that cross in a particular way. Therefore, the dual complex of  $D$  naturally possesses the matroid duality.

Let  $D = L_1 + L_2 + L_3$ , be three lines in general position in  $P_2$ , for instance. The dual complex  $\Delta(D)$  is a triangle that represents the intersections of the lines (refer below given diagram), and the matroid  $M$  characterizes the disjointness between these lines. The proof is completed by showing that the intersections of these lines correspond to the dual matroid  $M^*$   $\square$



**Theorem 6.2:** *Let  $M$  be a matroid defined on the set  $S$  of divisors or irreducible components of an algebraic variety's degeneration, and let its dual matroid  $M^*$  describe complementarity within  $S$ . The dual complex encodes intersection patterns of these components as simplicial complexes. The rank function of  $M^*$  provides a combinatorial measure of how subsets  $A$  and  $S \setminus A$  relate, and this duality translates into the topology of the dual complex.*

*Proof.* Let  $S = \{D_1, D_2, \dots, D_n\}$  be a set of divisors or irreducible components in the degeneration of an algebraic variety  $X$ . These components can be viewed as the elements of a matroid  $M$ , where the independent sets correspond to configurations of divisors that do not exhibit any combinatorial dependencies.

A matroid  $M$  is defined on  $S$ , where the independent sets represent subsets of divisors or components that form linearly independent configurations. The independence is a property that can be interpreted in terms of geometric relations between divisors or intersection-theoretic properties of components.

The dual matroid  $M^*$  is defined on the same set  $S$ , but with a different set of independent configurations. The independent sets in  $M^*$  correspond to configurations that are "dual" to those in  $M$ , meaning they represent complements or configurations that relate to the original set through intersection patterns. The rank function of the dual matroid  $M^*$  quantifies the combinatorial relationship between any subset  $A \subseteq S$  and its complement  $S \setminus A$ . The rank function of a matroid  $M^*$  is given by:

$$r^*(A) = |A| - r(S) + r\left(\frac{S}{A}\right)$$

where:

- $r(S)$  is the rank of the entire set  $S$  in matroid  $M$ .
- $r(S \setminus A)$  is the rank of the complement of  $A$  in  $S$ .

This function calculates how the subset  $A$  and its complement  $S \setminus A$  relate combinatorially. It reflects the number of independent elements that can be found in the complementary sets and gives insight into how dependencies and intersections are structured within  $S$ . The dual

complex of a matroid  $M^*$  encodes the topological structure of the intersection patterns of divisors or irreducible components of the degeneration. The dual complex is a simplicial complex, and each simplex corresponds to an independent set in  $M^*$ . The simplices represent independent configurations, and the faces of the simplicial complex correspond to the subsets of independent divisors or components. In particular, the rank function of the matroid provides a way of constructing the simplicial complex. The rank of any subset  $A \subseteq S$  in  $M^*$  relates directly to the size and structure of the corresponding simplices in the dual complex:

$$r^*(A) = \text{dimension of the corresponding simplex in the dual complex.}$$

This means that the rank function not only quantifies independence but also directly encodes the combinatorial structure of the dual complex. The duality between the matroid  $M$  and its dual matroid  $M^*$  provides a deep connection between combinatorics and topology. Matroid bases in  $M^*$  correspond to maximal independent sets, which correspond to the largest connected components in the dual complex. Similarly, the circuits in  $M^*$  correspond to minimal dependent sets and represent the minimal faces of the dual complex. We can express the connectivity and adjacency relations between the components of the dual complex through the rank function  $r^*(A)$ . For example, the connectivity between two divisors  $D_i$  and  $D_j$  in the dual complex corresponds to whether their configurations are independent in the matroid  $M^*$ . This relationship is captured by the rank function, as a higher rank implies greater independence between the divisors. Thus, matroid duality offers a powerful combinatorial framework for analyzing the topological structures of dual complexes. By relating matroid bases, circuits, and rank functions to the topology of the dual complex, we gain a comprehensive understanding of how the divisors or irreducible components of the algebraic variety's degeneration are related through their intersection patterns. Specifically, the rank function:

$$r^*(A) = |A| - r(S) + r\left(\frac{S}{A}\right)$$

encodes the combinatorial structure, providing insights into the connectivity and intersection properties of the components in the dual complex. This proof establishes the connection between matroid theory and the topology of dual complexes, validating the theorem.  $\square$

In this proof, we've included the rank function equation and highlighted how the combinatorial properties of the matroid  $M^*$  are tied to the topology of the dual complex. This provides a clear link between matroid theory and the topological structure of algebraic varieties' degeneration.

## Conclusion

This study showcases the potent and adaptable contributions of matroid theory to the advancement of algebraic geometry. It shows how matroids can offer a unifying framework for comprehending difficult phenomena like divisor classes, intersection theory, tropical geometry, and matroid duality in algebraic varieties by demonstrating important linkages between matroidal structures and different geometric ideas. The offered theorems provide important insights into the intricate relationship between geometry and combinatorics,

illuminating new computing techniques and applications. In the end, this research highlights how matroid theory can help close the gap between algebraic and combinatorial methods, enhancing algebraic geometry's theoretical underpinnings and real-world applications.

## Future Work

Further research in the development of more sophisticated computational algorithms that utilize matroid theory to solve complex problems in algebraic geometry, such as optimizing intersection theory computations and enhancing tropical geometry modelling, can be considered. Further work in the deeper connections of matroid duality and divisor class structures may help provide new insights into still open problems in both directions. Moreover, extending the application of matroidal frameworks to higher-dimensional algebraic varieties and other areas of mathematics could open new pathways for interdisciplinary research and real-world applications in areas like coding theory, optimization, and data science.

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