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ON THE SOLUTION EXISTENCE OF COUETTE FLOW MODEL FOR SWIRLING FLOWS

Gangipelli Chandrashekhar[1,*], V. Ganesh[2], R. Thenmozhy[3] and [4]M. Chandrashekharan.

[1]Department of Mathematics, University of Technology and Applied Sciences, Muscat, 133 Sultanate of Oman.
E-mail:chandu.724@gmail.com
[2]Department of Mathematics, University of Technology and Applied Sciences, Muscat, 133 Sultanate of Oman,
E-mail: profvganesh@gmail.com
[3]Department of Mathematics, Periyar Govt. Arts College, Cuddalore, India,
E-mail: thenmozhy17112002@gmail.com
[4]Department of Mathematics, University of Technology and Applied Sciences, Muscat, 133 Sultanate of Oman,
E-mail: chandrasekaran@utas.edu.om

*Corresponding author

Abstract We examine stability analysis of inviscid, hydromagnetic, incompressible flows with swirling motion. We derived general solution of the homogeneous problem in the case of Couette flow. We derived conditions for existence, non-existence of the series solutions for heterogeneous swirling flows. The results are depending on minimum Richardson number.

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1 Introduction

We study inviscid, hydromagnetic, coaxial flows which is also incompressible. The analysis of hydromagnetic, co-axial flows are important in understanding the mechanism in low-emission gas turbines [1], premixed swirl burners, rotating injectors [15], jet engine augmeters [4, 11]. When Michel Synge discriminant becomes zero, it leads to homogeneous case. [3] obtained a condition for instability. [3] obtained a semi-circle instability region. [14] obtained bounds for neutral phase speed. [14] obtained sufficient condition for stability. [16] derived parabolic instability regions under some condition. [5] derived instability region which intersect with [3]'s semi-circle under approximation. [6] derived a necessary for stability and derived unbounded instability region that intersect with [3]'s semicircle. [8] derived unbounded instability that intersect with [3]'s semicircle and obtained supremum for the amplification factor.

For hydromagnetic swirling flows [18] obtained a criterion for stability. [18] obtained a semi-elliptical bounded region with the condition on Richardson number. [17] derived instability regions under conditions. [14] derived a semi-elliptical bounded region under approximation that minimum curvature should be non-negative. [17] obtained estimate for growth rate. [7] developed a stability criterion based on the wave number. [7] obtained condition on wave number for existence of neutrally stable eigen solution. [9] obtained supremum's for the growth rate and bounds for the neutral phase speed. [10] obtained bounds for the complex phase speed and supremum's for the Growth rate. The spectrum of eigen values for spatially growing stability has been obtained by [12, 13]. When the heterogeneous problem has neutral mode with real phase velocity and there is a singular point then series solution needs to be obtained.

In this paper, we derived general series solution for circular Rayleigh problem and Heterogeneous differential equations for the Couette flow model. We derived the solutions of Heterogeneous differential equations in various cases, which depends up on Richardson number. Also, we derived the general solution of circular Rayleigh problem for Sinusoidal flow model.

2 HOMOGENEOUS PROBLEM

The homogeneous problem for the hydrodynamic stability is (cf. [2]),

$$\left(\frac{(ru)'}{r}\right)' - k^2u - \frac{r\left(\frac{W'}{r}\right)'u}{W-c} = 0, \tag{1}$$

with

$$u(r_1) = 0 = u(r_2). \tag{2}$$

Where k is the wave number,

c is complex eigen value,

$r\varepsilon[r_1,r_2]$, r is the radius of the cylinder.

$$u'' + \left(\frac{1}{r}\right)'u + \frac{1}{r}u' - k^2u - \left[\frac{W'' - \left(\frac{1}{r}\right)W'}{W-c}\right]u = 0. \tag{3}$$

Theorem 2.1: *If the Basic velocity function is $W = r$ and $r = r$ then the general series solution of circular Rayleigh problem is $u = c_1J_0(kr) + c_2Y_0(kr)$.*

Proof. If $W = r, r = r$.

(3) implies

$$u'' + \left(\frac{-1}{r^2}\right)u + \frac{1}{r}u' - k^2u - \left[\frac{1}{r(r-c)}\right]u = 0.$$

Since wave velocity $c = 0$,

$$r^2u'' + ru' - r^2k^2u = 0. \tag{4}$$

It can be converted in to a Bessel's equation using the transformation

$$u = U,\ r = \frac{R}{k}$$

and general solution is $u = c_1 J_0(kr) + c_2 Y_0(kr)$.

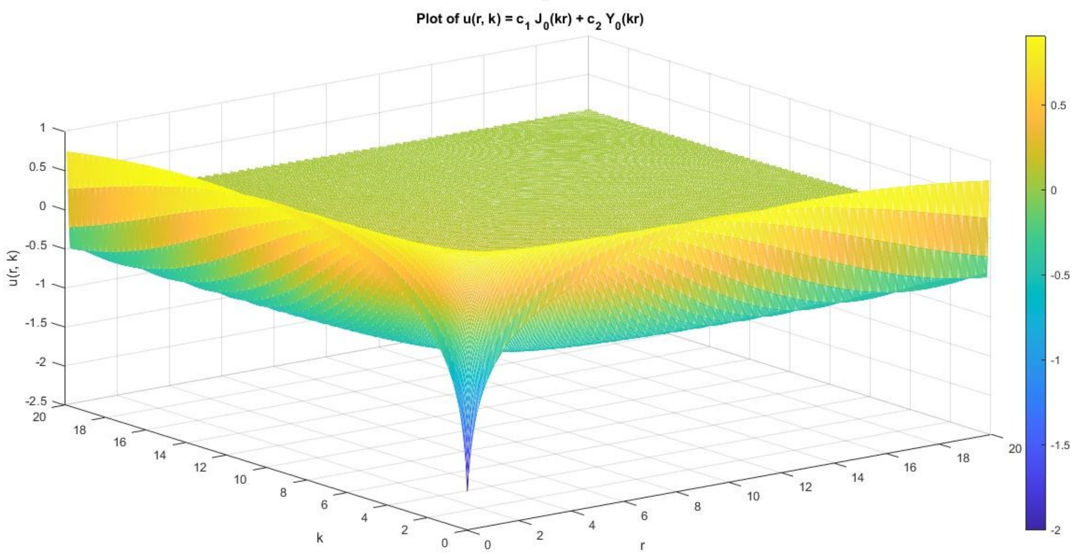

Figure 1: General solution of Theorem 2

Theorem 2.2: *If the Basic velocity function is* $W = \sin r^2$ *and* $r = r$ *then the general series solution of circular Rayleigh problem is*

$$u = \begin{Bmatrix} AY_1(4-k^2)r + BK_1(4-k^2)r & \text{when } k > 2 \\ AJ_1(4-k^2)r + BY_1(4-k^2)r & \text{when } k < 2 \end{Bmatrix}.$$

Proof. If $W = \sin r^2, r = r.$

$r\left(\frac{W'}{r}\right)' = 4r^2 \sin r^2$ and wave velocity $c = 0$, then

(3) implies

$$r^2u'' + ru' + (4r^2 - k^2r^2 - 1)u = 0.$$

It can be converted in to a Bessel's equation using the transformation

$$u = U, r = \frac{R}{\sqrt{\left|4-k^2\right|}}$$

and general solution is

$$u = \begin{Bmatrix} c_1 Y_1\left(4-k^2\right)r + c_2 K_1\left(4-k^2\right)r & when\ k>2 \\ c_1 J_1\left(4-k^2\right)r + c_2 Y_1\left(4-k^2\right)r & when\ k<2 \end{Bmatrix}.$$

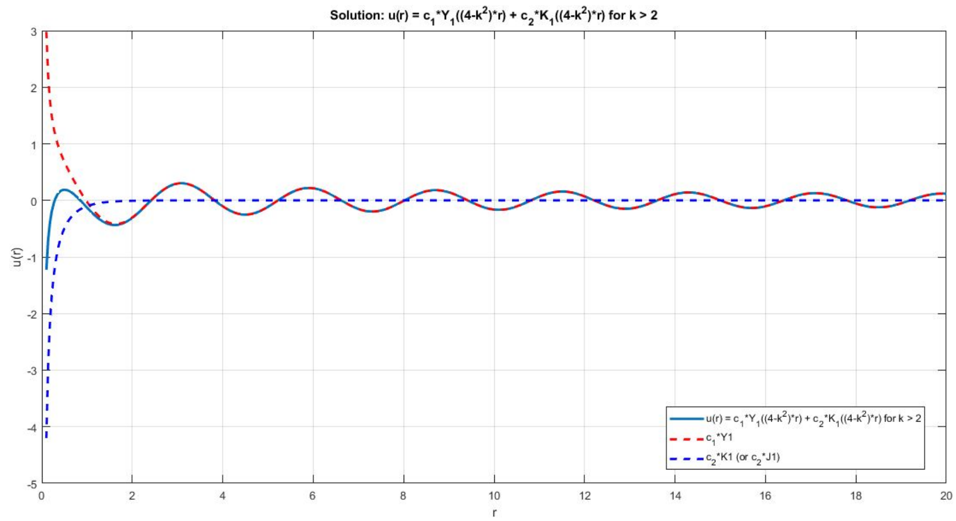

Figure 2: General solution of Theorem 1 for $k > 2$

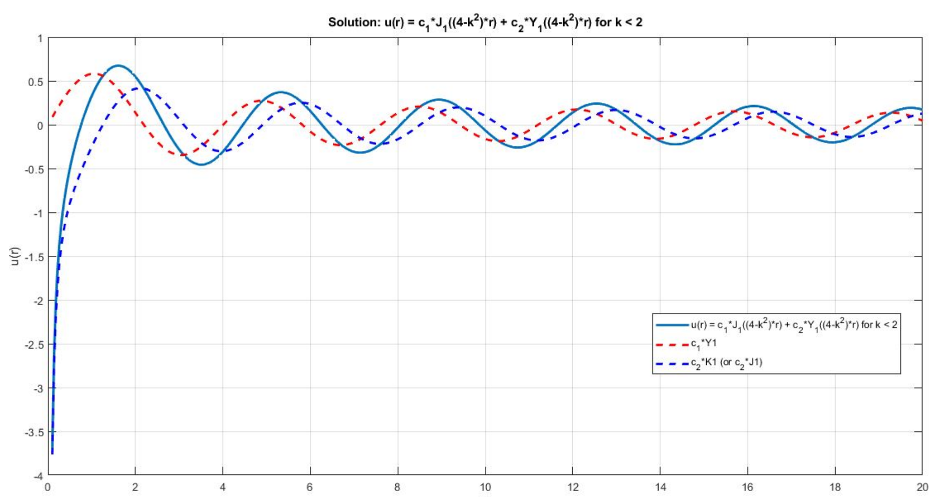

Figure 3: General solution of Theorem 1 for $k < 2$

3 NON-HOMOGENEOUS PROBLEM

The non-homogeneous differential equation is (cf. [18, 19])

$$\left(\frac{\rho_0(ru)'}{r}\right)' - k^2\rho_0 u - \frac{r\left(\frac{\rho_0 W'}{r}\right)' u}{W-c} + \frac{\psi u}{(W-c)^2} = 0, \tag{5}$$

with

$$u(r_1) = 0 = u(r_2) \tag{6}$$

Where μ magnetic permeability,

$\Omega_0 = \frac{V}{r}$ angular velocity and

$\psi = \frac{\rho_0 D\left(r^2\Omega_0\right)^2}{r^3} + (D\rho_0) r\Omega_0{}^2 - r\mu D\left(\frac{H_0^2}{r^2}\right)$ is Synge Michael discriminant.

$$u'' + \left(\frac{r'}{r}\right)' u + \frac{r'}{r}u' - k^2 u - \frac{r\left(\frac{W'}{r}\right)' u}{W-c} + \frac{\left(\frac{\psi}{\rho_0}\right)u}{(W-c)^2} = 0, \tag{7}$$

Let $r = r_c$ is a regular singular point, c is real and $W - c \neq 0$, if $c_i \neq 0$.
Using Frobenius expansion,

$$u = \sum_{m=0}^{\infty} a_m \left(r - r_c\right)^{m+r}.$$

When the Boussinesq approximation is applied, (7) implies,

$$u'' + \left(\frac{r'}{r}\right)' u + \frac{r'}{r}u' - \left(k^*\right)^2 u - \frac{r\left(\frac{W'}{r}\right)' u}{W-c^*} + \frac{\chi u}{\left(W-c^*\right)^2} = 0, \tag{8}$$

where $\chi = \frac{\psi}{\rho_0}$, ρ_0: nonzero constant,

$W(r^*)$: Unperturbed flow velocity,

$\chi(r^*)$: Unperturbed Micheal-Synge discriminant,

k^*: Wave number,

c^*: Wave velocity.

If $W\left(r_c^*\right) - c^* = 0,$

which implies $W\left(r_c^*\right)=c^*$. (critical layer)

Applying Taylor series expansion in (8), we have

$$\sum_{m=0}^{\infty} a_m\left(m+r\right)\left(m+r-1\right)\left(r-r_c\right)^{m+r-2}$$

$$+\left[\left(\frac{r'}{r}\right)'\left(r_c\right)+\frac{\left(\frac{r'}{r}\right)''\left(r_c\right)}{1!}\left(r-r_c\right)^1+\ldots.\right]\left[\sum_{m=0}^{\infty} a_m\left(r-r_c\right)^{m+r}\right]$$

$$+\left[\frac{r'\left(r_c\right)+\left[r''\left(r_c\right)-\frac{\left(r'\right)^2\left(r_c\right)}{r}\right]\left(r-r_c\right)+\ldots}{r\left(r_c\right)}\right]\left[\sum_{m=0}^{\infty} a_m\left(m+r\right)\left(r-r_c\right)^{m+r-1}\right]$$

$$-\left(k^*\right)^2\left[\sum_{m=0}^{\infty} a_m\left(r-r_c\right)^{m+r}\right]$$

$$-\left[\frac{r\left(\frac{W'}{r}\right)'+\left[\left[r\left(\frac{W'}{r}\right)'\right]'-\frac{W''r\left(\frac{W'}{r}\right)'}{2W'\left(r_c\right)}\right]\left(r-r_c\right)+\ldots}{W'\left(r-r_c\right)}\right]\left[\sum_{m=0}^{\infty} a_m\left(r-r_c\right)^{m+r}\right]$$

$$+\left[\frac{\chi\left(r_c\right)+\left[\chi'\left(r_c\right)-\frac{W''\left(r_c\right)\chi\left(r_c\right)}{W'\left(r_c\right)}\right]\left(r-r_c\right)+\ldots}{\left[W'\left(r_c\right)\left(r-r_c\right)\right]^2}\right]\left[\sum_{m=0}^{\infty} a_m\left(r-r_c\right)^{m+r}\right]=0. \tag{9}$$

Equating the least powers of $\left(r-r_c\right)$ to zero, $\left[\text{i.e, } \textit{powers of}\left(r-r_c\right)^{r-2}\right]$.

We have,

$$a_0 r\left(r-1\right)+a_0\frac{\chi(rc)}{\left[W'(rc)\right]^2}=0.$$

Since $a_0 \neq 0$, implies

$$r^2 - r + J = 0, \tag{10}$$

where $J = \dfrac{\chi(rc)}{[W'(rc)]^2}$ is called Richardson number.

Solving (10) we get,

$$r = \frac{1 \pm \sqrt{1-4J}}{2}.$$

$r = \dfrac{1}{2} \pm i\mu$, where $\mu = \sqrt{J - \dfrac{1}{4}}$.

We get, $R_+ = (W - c^*)^{\frac{1}{2}+i\mu}$, $R_- = (W - c^*)^{\frac{1}{2}-i\mu}$ are two linearly independent solutions of Eq.(5).

General solution of Eq.(5) is

$$u = AR_+ + BR_-,$$

i.e., $u = A(W - c^*)^{\frac{1}{2}+i\mu} + B(W - c^*)^{\frac{1}{2}-i\mu}$.

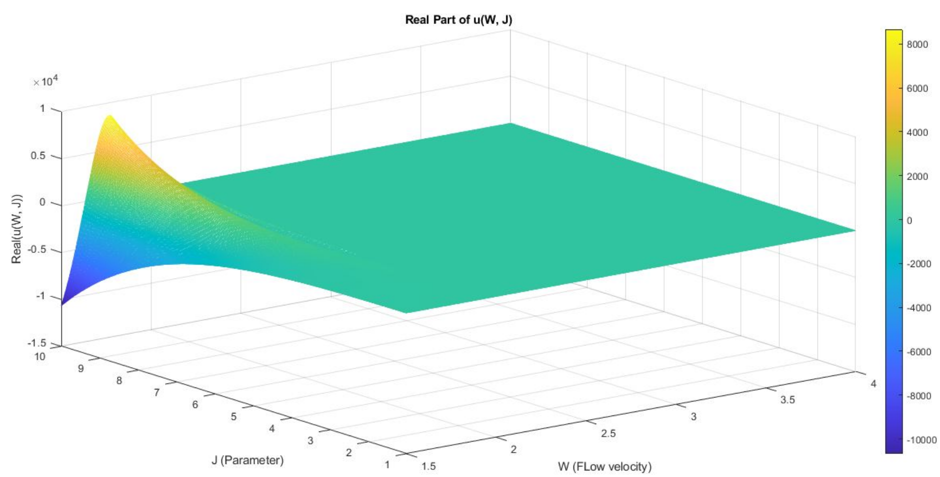

Figure 4: Real part to the solution of D.E. (5)

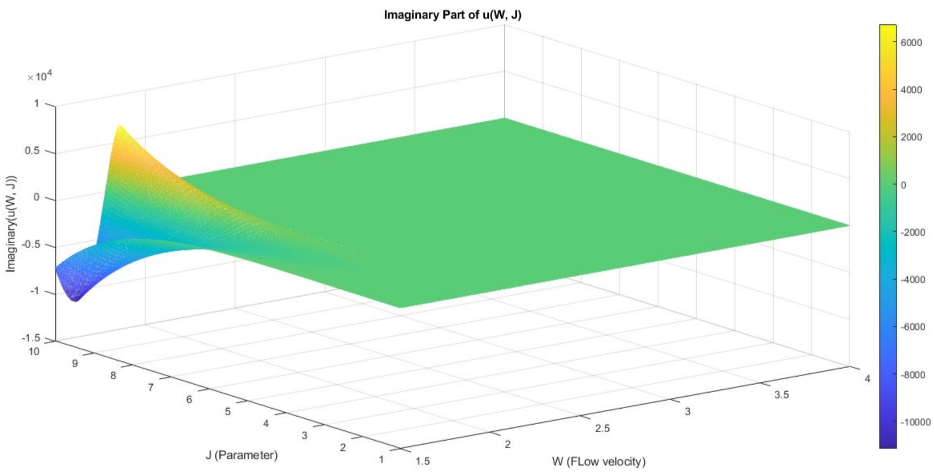

Figure 5: Imaginary part to the solution of D.E. (5)

Theorem 3.1: *If the Richardson number* $J = \frac{1}{4}$ *then solution of (5) is*

$$u = \left(W - c^*\right)^{\frac{1}{2}} r_+.$$

Proof. If $J = \frac{1}{4}$, then $\mu = 0$.

From (9) we have, $r = \frac{1}{2}, \frac{1}{2}$.

Which implies, the solution of (5) is

$$u = \left(W - c^*\right)^{\frac{1}{2}} r_+.$$

Theorem 3.2: *If the Richardson number* $0 < J < \frac{1}{4}$ *then solution of (5) is either*

$$u = \left(W - c^*\right)^{\frac{1}{2}+\mu} r_+ \ \textit{or} \ \ u = \left(W - c^*\right)^{\frac{1}{2}-\mu} r_-.$$

Proof. If $0 < J < \frac{1}{4}$ then $J - \frac{1}{4} > 0$.

From (9) we have, $r_{1,2} = \frac{1}{2} \pm \mu$.

Which implies, the solution of (5) is either

$u = \left(W - c^*\right)^{\frac{1}{2}+\mu} r_+$ or $u = \left(W - c^*\right)^{\frac{1}{2}+\mu} r_-$.

Theorem 3.3: *If the Richardson number* $J=0, a_1 \neq 0$, *then solution of (5) must be proportional to* $u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_+$ *and if* $a_1=0$, *then the two linearly independent solutions of (5) are* $u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_+$ *and* $u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_-$.

Proof. If $J=0$, then we have to consider two cases depending on how the expression in the brackets in (9) behaves near the critical layer $c^*=W\left(r_c^*\right)$.

The two cases are occur when $a_1 \neq 0$ and $a_1=0$.

Case (i) If $a_1 \neq 0$ then there exit a singular solution which is proportional to $u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_+$.

Case (ii) If $a_1=0$ from (9) we have two linearly independent solutions.

$u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_+$ and $u=\left(W-c^*\right)^{\frac{1}{2}+\mu} r_-$.

Theorem 3.4: *If the Richardson number* $J>\frac{1}{4}$ *then there is no singular mode solution of (5).*

Proof. If $J>\frac{1}{4}$, then there is singular solution because $c_i^* \neq 0$.

$$\begin{pmatrix} Since\, c=c_r+ic_i \\ c^*=c_r-ic_i \text{ and } -c_i<0. \end{pmatrix}$$

4 CONCLUDING REMARKS

In the present work, we study inviscid, incompressible, swirling flows. When magnetic term remains zero it reduces to circular Rayleigh problem. For this, we obtained the general solution in cases of Couette, Sinusoidal flows considered as basic flow model using Bessel's function. For the case of Heterogeneous swirling flows, we obtained condition for existence or non-existence of series solution, for the Couette flow model. The series solution depends on Richardson.

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