NEW FORM OF SEPARATION AXIOMS IN FUZZY SOFT SEQUENTIAL TOPOLOGICAL SPACES

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Abstract: The objective of this paper is to explore separation axioms in fuzzy soft sequential topological spaces. In 2002, Bose and Indrajit Lahiri [2] gave the concept of separation axioms in sequential topological spaces. In 2018**,** Mahanta and Das [7] introduced fuzzy soft point and studied separation axioms in fuzzy soft topological spaces. Khedr et al [6] introduced fuzzy soft separation axioms in terms of the modified definitions of the fuzzy soft point and study some of the properties in the year 2019. By the way of above concepts, we introduce the separation axioms in a new way in fuzzy soft sequential topology. We define fuzzy soft sequential T_i axioms ($i = 0,1,2$) using the concept of fuzzy soft sequential quasi-coincident relation between fuzzy soft sequential set and fuzzy soft sequential point and discuss some properties of these spaces with proper examples in detail.

Keywords: fuzzy soft set, fuzzy soft sequential set, fuzzy soft sequential point, $FSS - T_0$ space, $FSS - T_1$ space, $FSS - T_2$ space.

1. Introduction

In 1965, the fuzzy set was introduced by **L. A. Zadeh** [15], which is dealing with uncertainty. In 1999, **Molodtsov** [9] introduced the concept of soft set theory, which provides a new mathematical theory for dealing with uncertainty. In 2001, **Maji et al** [8] proposed the concept of fuzzy soft set which is a new mathematical approach to vagueness by involving the ideas of both fuzzy set and soft set. In 2002, **Bose and Indrajit Lahiri** [2] introduced the concept of sequential topological spaces. He defined any sequence of subsets of a non void set is called a sequential set and proposed properties of sequential set, sequential topology, separation axioms in detail. Fuzzy soft set was further improved by **Ahmad and Kharal** [1] in the year 2009. In the same year, **Tanay and Kandemir** [13] introduced the topological structure of fuzzy soft sets and studied some of its structural properties. In continuation, **Mahanta and Das** [7] introduced fuzzy soft point and explored some concepts in fuzzy soft topological spaces in the year 2018. Further, he presented separation axioms and investigated for fuzzy soft topological spaces. In 2019, **Khedr et al** [6] introduced fuzzy soft separation axioms in terms of the modified definitions of the fuzzy soft point and studied some of the properties. In 2024, we have introduced fuzzy soft sequential set [3] by merging the concept of fuzzy soft set and sequential set. In continuation, we extend the concept into fuzzy soft sequential topological spaces [4] and fuzzy soft sequential point [5] are explored in detail.

In this paper, we propose separation axioms in a different way in fuzzy soft sequential topology. We define fuzzy soft sequential T_i axioms $(i = 0,1,2)$ using the concept of fuzzy soft sequential quasi-coincident relation via fuzzy soft sequential set and fuzzy soft sequential point and analyze some properties of these spaces with needed examples in detail.

2.Preliminaries

Definition 2.1 [15] A fuzzy set X over a universal set U is a set defined by a function μ_x performing a mapping $\mu_x : U \to [0,1]$, here this μ_x is the membership function of X, and the value $\mu_x(u)$ will be the grade of membership of u $\in U$.

Definition 2.2 [9] Let U be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U.

Definition 2.3 [8] Let $\wp(U)$ denotes the set of all fuzzy sets of U. Let $A_i \subseteq E$, a set of parameters. A pair (F_i, A_i) is called a fuzzy soft set over U, where F_i is a mapping given by $F_i: A_i \to \wp(U)$.

Definition 2.4 [13] Let (γ, X) be an element of $\mathcal{F}\mathcal{s}(U;E), \mathcal{P}(\gamma, X)$ be the set of all fuzzy soft subsets of (γ, X) and $\tilde{\tau}$ be a subfamily of $\mathcal{P}(\gamma, X)$. Then $\tilde{\tau}$ is called fuzzy soft topology on (γ, X) if the following conditions are satisfied:

i. $\widetilde{\emptyset}_X , (\gamma, X) \in \tilde{\tau}$
ii. $(f, A) , (g, B) \in \tilde{\tau} \Rightarrow$
$(f, A) \tilde{\cap} (g, B) \in \tilde{\tau}$

iii. $\{(f,A)_k | k \in K\} \subset \tilde{\tau} \Rightarrow \tilde{\bigcup}_{k\in K}(f,A)_k \in \tilde{\tau}$.

The pair $(X_\gamma, \tilde{\tau})$ is called a fuzzy soft topological space.

Definition 2.5 [10] A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$ we denote this fuzzy point by x_λ, where the point x is called its support.

Definition 2.6 [12] The fuzzy soft set $f_A \in FS(X,E)$ is called fuzzy soft point, if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X. That is, there exists $x \in X$ such that $f_A(e)(x) = \alpha (0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denote this fuzzy soft point $f_A = e_x^\alpha = \{(e, x_\alpha)\}$.

Definition 2.7 [7] A fuzzy soft topological space (f_A,τ) is said to be a fuzzy soft $T_0 -$ space if for every pair of disjoint fuzzy soft points e_{h_A}, e_{g_B}, there exist fuzzy soft open set containing one but not the other.

Definition 2.8 [7] A fuzzy soft topological space (f_A,τ) is said to be a fuzzy soft $T_1 -$ space if for distinct pair of fuzzy soft points e_{g_A}, e_{k_A} of f_A, there exist fuzzy soft open sets s_A and h_A such that $e_{g_A} \tilde{\in} s_A$ and $e_{g_A} \tilde{\notin} h_A$; $e_{k_A} \tilde{\in} h_A$ and $e_{k_A} \tilde{\notin} s_A$.

Definition 2.9 [7] A fuzzy soft topological space (f_A,τ) is said to be a fuzzy soft $T_2 -$ space if and only if for distinct fuzzy soft points e_{g_A}, e_{k_A} of f_A, there exist disjoint fuzzy soft open sets h_A and s_A such that $e_{g_A} \tilde{\in} h_A$ and $e_{k_A} \tilde{\in} s_A$.

Definition 2.10 [2] Any sequence of subsets of a non void set X is called a sequential set in X. That is, $A(s) = \{A_n\}_{n=1}^{\infty}$, where each A_n is a subset of X, is a sequential set in X. The subsets $A_n, n \in \mathbb{N}$ are called the components of $A(s)$.

Definition 2.11 [2] Let X be a non void set. A collection τ of sequential sets in X is said to form a sequential topology on X if

i. $\emptyset(s), X(s) \in \tau$.
ii. Arbitrary union of members of τ is a member of τ.
iii. Finite intersection of members of τ is a member of τ.

A set X equipped with a sequential topology τ is called a sequential topological space, denoted by (X, τ). The members of τ are called $\tau -$ open sequential sets in X.

Definition 2.12 [2] A sequential point p is a sequential set $P(s) = \{P_n\}_{n=1}^{\infty}$ with support $x \in X$ and base $M \subset \mathbb{N}$ if

$$P_n = x \quad \forall\, n \in M,$$

$$= \emptyset \quad \forall\, n \in \mathbb{N} - M$$

and write $p = (x, M)$.

Definition 2.13 [2] A sequential topological space (X, τ) is said to be T_0 space if for any two distinct sequential points $p = (x, P)$ and $q = (y, Q)$ there exist an open sequential set $U(s)$ in (X, τ) such that $p \in_w U(s)$ and $q \notin_w U(s)$.

Definition 2.14 [2] A sequential topological space (X, τ) is said to be T_1 space if for any two distinct sequential points $p = (x, P)$ and $q = (y, Q)$ there exist an open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w U(s)$, $q \in_w V(s)$, $p \notin_w V(s)$ and $q \notin_w U(s)$.

Definition 2.15 [2] A sequential topological space (X, τ) is said to be Hausdorff or T_2 space if for any two distinct sequential points $p = (x, P)$ and $q = (y, Q)$ there exist an open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w U(s)$, $q \in_w V(s)$, $p \notin_w \overline{V(s)}$ and $q \notin_w \overline{U(s)}$.

Definition 2.16 [3] A sequence of fuzzy soft sets is a mapping from $\mathbb{N}$ to the family of all fuzzy soft sets and is denoted by $\{(F, A)_n\}$ or $\{(F, A)_n; n = 1,2, \dots\}$.That is, $\{(F, A)_n, n \in \mathbb{N}\}$ where $(F, A)_n$ for each $n \in \mathbb{N}$ represents components of fuzzy soft set in $\{(F, A)_n\}$ and $n \in \mathbb{N}$, the set of all natural numbers. A sequence of fuzzy soft sets is called fuzzy soft sequential set.

Definition 2.17 [4] A family $\tilde{\tau}$ of fuzzy soft sequential sets on U satisfying the properties:

(i) $0^{\mathbb{N}}, 1^{\mathbb{N}} \in \tilde{\tau}$

(ii) $\{(F, A)_n\}$, $\{(G, B)_n\} \in \tilde{\tau} \Rightarrow \{(F, A)_n\} \tilde{\cap} \{(G, B)_n\} \in \tilde{\tau}$ and

(iii) For any family $\{\{(F_\lambda, A)_n\}, \lambda \in \Lambda\} \in \tilde{\tau} \Rightarrow \tilde{\bigcup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}$

is called a fuzzy soft sequential topology on U and the triplet $(U, \tilde{\tau}, E)$ is called a fuzzy soft sequential topological space over U.

Definition 2.18 [5] A fuzzy soft sequential set is said to be a fuzzy soft sequential point (briefly, FSS-point) is denoted by $\{(P^n_{e_i x_j}, r)_E\}$, if there exists $e_i \in A \subseteq E$ and $x_j \in U$ for fixed i and j, $P_{\mathrm{n}}(e_i)(x_j)$ is a fuzzy point for all $n \in \mathbb{N}$. That is, there exists $e_i \in A \subseteq E, x_j \in U$ for fixed i and j, and a nonzero sequence r = $\{r_n\}$ for all $n \in \mathbb{N}$ in I such that,

$$P_{\mathrm{n}}(e_i)(y_j) = \begin{cases} r_n, & if\ y_j = x_j \\ 0, & if\ y_j \in U - \{x_j\} \end{cases}$$

Where $P_{\mathrm{n}}(e_i)(y_j)$ be the membership value of $\{(P^n_{e_i x_j}, r)_E\}$.

Definition 2.19 [5] A fuzzy soft sequential point $\{(P^n_{e_i x_j}, r)_E\}$ is called fuzzy soft sequential quasi-coincident with the fuzzy soft sequential set $\{(F,A)_n\}$, denoted by $\{(P^n_{e_i x_j}, r)_E\} q \{(F,A)_n\}$, if there exists $e_i \in A$ and $x_j \in U$ for fixed i and j such that $P_n(e_i)(x_j) + F_n(e_i)(x_j) \gtrsim 1$ (or) $P_n(e_i)(x_j) \gtrsim F^c_n(e_i)(x_j)$ for all $n \in \mathbb{N}$.

If there exists $e_i \in A$ and $x_j \in U$ for fixed i and j such that $P_n(e_i)(x_j) + F_n(e_i)(x_j) \lesseqqgtr 1$ for all $n \in \mathbb{N}$, then $\{(P^n_{e_i x_j}, r)_E\}$ and $\{(F,A)_n\}$ are not fuzzy soft sequential quasi-coincident and is denoted by $\{(P^n_{e_i x_j}, r)_E\} \bar{q} \{(F,A)_n\}$.

Definition 2.20 [5] A fuzzy soft sequential point $\{(P^n_{e_i x_j}, r)_E\}$ is called fuzzy soft sequential weakly quasi-coincident with the fuzzy soft sequential set $\{(F,A)_n\}$, denoted by $\{(P^n_{e_i x_j}, r)_E\} q_w \{(F,A)_n\}$, if there exists $e_i \in A$ and $x_j \in U$ for fixed i and j such that $P_n(e_i)(x_j) + F_n(e_i)(x_j) \gtrsim 1$ (or) $P_n(e_i)(x_j) \gtrsim F^c_n(e_i)(x_j)$ for some $n \in \mathbb{N}$.

If $\{(P^n_{e_i x_j}, r)_E\}$ and $\{(F,A)_n\}$ are not fuzzy soft sequential weakly quasi-coincident, then it can be written as $\{(P^n_{e_i x_j}, r)_E\} \bar{q}_w \{(F,A)_n\}$.

3. Main Results

3.1 Fuzzy soft sequential T_0 space

Definition 3.1.1 A fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$ is said to be **fuzzy soft sequential T_0 space (briefly, FSS - T_0 space)** if for any two distinct fuzzy soft sequential points $\{(P^n_{e_i x_j}, r)_E\}$ and $\{(Q^n_{e_i x_j}, s)_E\}$, there exists a fuzzy soft sequential open set $\{(F,A)_n\}$ such that $\{(P^n_{e_i x_j}, r)_E\}\, q_w \{(F,A)_n\}$ and $\{(Q^n_{e_i x_j}, s)_E\} \bar{q}_w \{(F,A)_n\}$.

Example 3.1.2 Let us consider the universe $U = \{x_1, x_2\}$, E={ e_1, e_2, e_3} and $A = \{e_1, e_2\} \subseteq E$ be a collection of sets of parameters. We define $\{(P^n_{e_i x_j}, r)_E\}$ and $\{(Q^n_{e_i x_j}, s)_E\}$ are distinct fuzzy soft sequential points.

Consider $\{(P^n_{e_i x_j}, r)_E\} = \left\{\left(e_1, \left\{\frac{x_1}{(n/n+7)}, \frac{x_2}{(0)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(0)}, \frac{x_2}{(0)}\right\}\right)\right\} \forall\, n \in \mathbb{N}$

and $\{(Q^n_{e_i x_j}, s)_E\} = \left\{\left(e_1, \left\{\frac{x_1}{(n/2n+1)}, \frac{x_2}{(0)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(0)}, \frac{x_2}{(0)}\right\}\right)\right\} \forall\, n \in \mathbb{N}$

Let us consider a fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$, $\tilde{\tau} = \{0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(G,A)_n\}, \{(F,A)_n\}\}$, where

$$\{(G,A)_n\} = \left\{\left(e_1, \left\{\frac{x_1}{(0.15)}, \frac{x_2}{(1/7n)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(1/5n)}, \frac{x_2}{(1/3n)}\right\}\right)\right\} \forall\, n \in \mathbb{N}$$

and $\{(F,A)_n\} = \left\{\left(e_1,\left\{\frac{x_1}{(0.4)},\frac{x_2}{(1/n)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(0.3)},\frac{x_2}{(\sqrt{3}/2n)}\right\}\right)\right\}\ \forall\, n \in \mathbb{N}$

From the above distinct fuzzy soft sequential points, there exists $e_1 \in A, x_1 \in U$ such that $P_{\mathrm{n}}(e_1)(x_1) + G_{\mathrm{n}}(e_1)(x_1) \gtrapprox 1$ for some $n \in \mathbb{N}$ and $Q_{\mathrm{n}}(e_1)(x_1) + G_{\mathrm{n}}(e_1)(x_1) \lneqq 1$ for all $n \in \mathbb{N}$. Hence there exists a fuzzy soft sequential open set $\{(G,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\, q_w\{(G,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\, \bar{q}_w\{(G,A)_n\}$.

Similarly, there exists $e_1 \in A, x_1 \in U$ such that $P_{\mathrm{n}}(e_1)(x_1) + F_{\mathrm{n}}(e_1)(x_1) \gtrapprox 1$ for some $n \in \mathbb{N}$ and $Q_{\mathrm{n}}(e_1)(x_1) + F_{\mathrm{n}}(e_1)(x_1) \lneqq 1$ for all $n \in \mathbb{N}$. Hence there exists a fuzzy soft sequential open set $\{(F,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\, q_w\{(F,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\, \bar{q}_w\{(F,A)_n\}$.

Hence $(U,\tilde{\tau},E)$ is FSS - T_0 space.

Note 3.1.3. The fuzzy soft sequential indiscrete topology is not a FSS - T_0 space.

Theorem 3.1.4 A fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ is a FSS - T_0 space if and only if for any two distinct fuzzy soft sequential points $\{(P^n_{e_ix_j},r)_E\}$ and $\{(Q^n_{e_ix_j},s)_E\}$ either $\{(P^n_{e_ix_j},r)_E\}$ does not belong to the closure of $\{(Q^n_{e_ix_j},s)_E\}$ or $\{(Q^n_{e_ix_j},s)_E\}$ does not belong to the closure of $\{(P^n_{e_ix_j},r)_E\}$.

Proof.

Necessity: Let $(U,\tilde{\tau},E)$ be a FSS - T_0 space. By Definition 3.1.1, for a distinct fuzzy soft sequential points $\{(P^n_{e_ix_j},r)_E\}$ and $\{(Q^n_{e_ix_j},s)_E\}$ there exists a fuzzy soft sequential open set $\{(F,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\, q_w\{(F,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\bar{q}_w\{(F,A)_n\}$. Therefore, $\{(P^n_{e_ix_j},r)_E\} \notin \overline{\{(Q^n_{e_ix_j},s)_E\}}$.

Sufficiency: Assume $\{(P^n_{e_ix_j},r)_E\}$ does not belong to the closure of $\{(Q^n_{e_ix_j},s)_E\}$ and both fuzzy soft sequential points are distinct. Suppose $(U,\tilde{\tau},E)$ is not FSS - T_0 space. Then there exists a fuzzy soft sequential open set $\{(F,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\, q_w\{(F,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}q_w\{(F,A)_n\}$. By our assumption, $\{(Q^n_{e_ix_j},s)_E\}\bar{q}_w\{(F,A)_n\}$, which is a contradiction. Then there exists $\{(F,A)_n\} \in \tilde{\tau}$ such that $\{(P^n_{e_ix_j},r)_E\}\, q_w\{(F,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\bar{q}_w\{(F,A)_n\}$. Hence $(U,\tilde{\tau},E)$ is FSS - T_0 space.

Corollary 3.1.5 A fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ is FSS - T_0 space if and only if any two distinct fuzzy soft sequential points have distinct fuzzy soft sequential closure.

Proof. Assume $(U,\tilde{\tau},E)$ is FSS - T_0 space. $\Leftrightarrow$ By Theorem 3.1.4, any two distinct fuzzy soft sequential points have distinct fuzzy soft sequential closure.

3.2 Fuzzy soft sequential T_1 space

Definition 3.2.1 A fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ is said to be **fuzzy soft sequential T_1 space (briefly, FSS - T_1 space)** if for any two distinct fuzzy soft sequential points $\{(P^n_{e_i x_j}, r)_E\}$ and $\{(Q^n_{e_i x_j}, s)_E\}$, there exist fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(P^n_{e_i x_j}, r)_E\} q_w \{(F,A)_n\}$, $\{(Q^n_{e_i x_j}, s)_E\}\ q_w \{(G,A)_n\}$, $\{(P^n_{e_i x_j}, r)_E\} \bar{q}_w \{(G,A)_n\}$, $\{(Q^n_{e_i x_j}, s)_E\}\ \bar{q}_w \{(F,A)_n\}$.

In other words, a fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ is said to be FSS - T_1 space if every fuzzy soft sequential point over U is fuzzy soft sequential closed.

Example 3.2.2 Let us consider the universe $U=\{x_1,x_2\}$, E=\{ e_1, e_2, e_3\} and $A=\{e_1,e_2\}\subseteq E$ be a collection of sets of parameters. The distinct fuzzy soft sequential points are defined as follows:

Consider $\{(P^n_{e_i x_j}, r)_E\} = \left\{\left(e_1,\left\{\frac{x_1}{(1/7n)},\frac{x_2}{(0)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(0)},\frac{x_2}{(0)}\right\}\right)\right\}\ \forall\, n\in \mathbb{N}$

and $\{(Q^n_{e_i x_j}, s)_E\} = \left\{\left(e_1,\left\{\frac{x_1}{(0)},\frac{x_2}{(0)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(\sqrt{3}/2n)},\frac{x_2}{(0)}\right\}\right)\right\}\ \forall\, n\in \mathbb{N}$

First Part: Let us consider a fuzzy soft sequential topological space $(U,\tilde{\tau},E)$, $\tilde{\tau}=\{0^{\mathbb{N}},1^{\mathbb{N}},\{(G,A)_n\},\{(F,A)_n\}\}$, where

$$\{(G,A)_n\} = \left\{\left(e_1,\left\{\frac{x_1}{(0.15)},\frac{x_2}{(1/7n)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(1/5n)},\frac{x_2}{(1/3n)}\right\}\right)\right\}\ \forall\, n\in \mathbb{N}$$

And $\{(F,A)_n\} = \left\{\left(e_1,\left\{\frac{x_1}{(0.9)},\frac{x_2}{(1/n)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(0.5)},\frac{x_2}{(\sqrt{3}/2n)}\right\}\right)\right\}\ \forall\, n\in \mathbb{N}$

For the fuzzy soft sequential point $\{(P^n_{e_i x_j}, r)_E\}$, there exists $e_1\in A, x_1\in U$ such that $P_{\mathrm{n}}(e_1)(x_1)+F_{\mathrm{n}}(e_1)(x_1) \gtrsim 1$ for some $n\in\mathbb{N}$ and $P_{\mathrm{n}}(e_1)(x_1)+G_{\mathrm{n}}(e_1)(x_1) \lesseqqgtr 1$ for all $n\in\mathbb{N}$. That is, there exists fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(P^n_{e_i x_j}, r)_E\}\ q_w \{(F,A)_n\}$ and $\{(P^n_{e_i x_j}, r)_E\}\ \bar{q}_w \{(G,A)_n\}$.

Similarly, for the fuzzy soft sequential point $\{(Q^n_{e_i x_j}, s)_E\}$, there exists fuzzy soft sequential open sets $\{(F,A)_n\}$ and

$\{(G,A)_n\}$ such that $\{(Q^n_{e_ix_j},s)_E\}\ q_w\{(G,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\ q_w\{(F,A)_n\}$.

Since $\{(Q^n_{e_ix_j},s)_E\}\ q_w\{(F,A)_n\}$, $(U,\tilde{\tau},E)$ is not FSS - T_1 space.

Second Part: If we consider $\{(F,A)_n\} = \left\{\left(e_1,\left\{\frac{x_1}{(0.9)},\frac{x_2}{(1/n)}\right\}\right),\left(e_2,\left\{\frac{x_1}{(1)},\frac{x_2}{(\sqrt{3}/2n)}\right\}\right)\right\}\ \forall\, n\in\mathbb{N}$ in the first part, we get $\{(Q^n_{e_ix_j},s)_E\}\ \bar{q}_w\{(F,A)_n\}$. Hence $(U,\tilde{\tau},E)$ is FSS - T_1 space.

Note 3.2.3 The fuzzy soft sequential indiscrete topology is not a FSS - T_1 space.

Theorem 3.2.4 Every FSS - T_1 space is FSS - T_0 space.

Proof. Assume that $(U,\tilde{\tau},E)$ is FSS - T_1 space. Then for distinct fuzzy soft sequential points $\{(P^n_{e_ix_j},r)_E\}$ and $\{(Q^n_{e_ix_j},s)_E\}$, there exists fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\ q_w\{(F,A)_n\}$, $\{(Q^n_{e_ix_j},s)_E\}q_w\{(G,A)_n\}$, $\{(P^n_{e_ix_j},r)_E\}\ \bar{q}_w\{(G,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\ \bar{q}_w\{(F,A)_n\}$. Using Definition 3.1.1, $(U,\tilde{\tau},E)$ is FSS - T_0 space.

The converse of the above theorem need not be true is shown by the following example.

Example 3.2.5 Example 3.1.2 shows that, $\{(P^n_{e_ix_j},r)_E\}q_w\ \{(G,A)_n\}$ and $\{(Q^n_{e_ix_j},s)_E\}\ \bar{q}_w\{(G,A)_n\}$. But $\{(Q^n_{e_ix_j},s)_E\}\bar{q}_w\{(F,A)_n\}$ and $\{(P^n_{e_ix_j},r)_E\}\ q_w\{(F,A)_n\}$ implies $(U,\tilde{\tau},E)$ is not FSS - T_1 space.

3.3 Fuzzy soft sequential T_2 - space or Fuzzy soft sequential Hausdorff space

Definition 3.3.1 A fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ is said to be **fuzzy soft sequential Hausdorff space (briefly, FSS - T_2 space)** if for any two distinct fuzzy soft sequential points $\{(P^n_{e_ix_j},r)_E\}$ and $\{(Q^n_{e_ix_j},s)_E\}$, there exist fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(P^n_{e_ix_j},r)_E\}\ q_w\{(F,A)_n\}$, $\{(Q^n_{e_ix_j},s)_E\}q_w\{(G,A)_n\}$, $\{(P^n_{e_ix_j},r)_E\}\ \bar{q}_w\overline{\{(G,A)_n\}}$ and $\{(Q^n_{e_ix_j},s)_E\}\bar{q}_w\overline{\{(F,A)_n\}}$.

Example 3.3.2 Let us consider the universe $U=\{x_1,x_2\}$, E={ e_1, e_2, e_3} and $A=\{e_1,e_2\}\subseteq E$ be a collection of sets of parameters. The fuzzy soft sequential points $\{(P^n_{e_ix_j},r)_E\}$ and $\{(Q^n_{e_ix_j},s)_E\}$ are defined in Example 3.2.2.

From the second part of the Example 3.2.2, $\{(P^n_{e_ix_j}, r)_E\}\, q_w\, \{(F,A)_n\}$, $\{(Q^n_{e_ix_j}, s)_E\}\, q_w\, \{(G,A)_n\}$.

Claim: $\{(P^n_{e_ix_j}, r)_E\}\, \bar{q}_w \overline{\{(G,A)_n\}}$, $\{(Q^n_{e_ix_j}, s)_E\}\, \bar{q}_w \overline{\{(F,A)_n\}}$.

Some of the fuzzy soft sequential closed sets are $0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(G,A)^c_n\}, \{(F,A)^c_n\}$.

Here $\overline{\{(F,A)_n\}} = 1^{\mathbb{N}}$ and $\overline{\{(G,A)_n\}}$ = $\tilde{\cap}\,\{1^{\mathbb{N}}, \{(G,A)^c_n\}\} = \{(G,A)^c_n\}$.

Therefore, there exists $e_1 \in A, x_1 \in U$ such that $P_n(e_1)(x_1) + G^c_n(e_1)(x_1) \lneqq 1$ for all $n \in \mathbb{N}$ implies $\{(P^n_{e_ix_j}, r)_E\}\, \bar{q}_w \overline{\{(G,A)_n\}}$. Also, $\{(Q^n_{e_ix_j}, s)_E\}\, \bar{q}_w \overline{\{(F,A)_n\}}$. Hence $(U, \tilde{\tau}, E)$ is FSS - T_2 space.

Note 3.3.3 The fuzzy soft sequential indiscrete topology is not a FSS - T_2 space.

Theorem 3.3.4 Every FSS - T_2 space is FSS - T_1 space.

Proof. Assume $(U, \tilde{\tau}, E)$ is FSS - T_2 space. By Definition 3.3.1, for distinct fuzzy soft sequential points $\{(P^n_{e_ix_j}, r)_E\}$ and $\{(Q^n_{e_ix_j}, s)_E\}$, there exists fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(P^n_{e_ix_j}, r)_E\}\, q_w\, \{(F,A)_n\}$, $\{(Q^n_{e_ix_j}, s)_E\} q_w\, \{(G,A)_n\}$, $\{(P^n_{e_ix_j}, r)_E\}\, \bar{q}_w \overline{\{(G,A)_n\}}$, $\{(Q^n_{e_ix_j}, s)_E\}\, \bar{q}_w \overline{\{(F,A)_n\}}$. Since $\overline{\{(G,A)_n\}}$ and $\overline{\{(F,A)_n\}}$ is the smallest fuzzy soft sequential closed set containing $\{(G,A)_n\}$ and $\{(F,A)_n\}$, respectively. Then $\{(P^n_{e_ix_j}, r)_E\}\bar{q}_w\{(G,A)_n\}$, $\{(Q^n_{e_ix_j}, s)_E\}\, \bar{q}_w\{(F,A)_n\}$. Hence $(U, \tilde{\tau}, E)$ is FSS - T_1 space.

The converse of the above theorem need not be true is shown by the following example.

Example 3.3.5 Let us consider the universe $U = \{x_1, x_2\}$, E={ e_1, e_2, e_3} and $A = \{e_1, e_2\} \subseteq E$ be a collection of sets of parameters. Consider the fuzzy soft sequential points $\{(P^n_{e_ix_j}, r)_E\}$ and $\{(Q^n_{e_ix_j}, s)_E\}$ was given in Example 3.2.2 and also consider a fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$, $\tilde{\tau} = \{0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(G,A)_n\}, \{(F,A)_n\}\}$, where

$$\{(G,A)_n\} = \left\{\left(e_1, \left\{\frac{x_1}{(0.1)}, \frac{x_2}{(1/7n)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(0.2)}, \frac{x_2}{(1/5n)}\right\}\right)\right\} \forall\, n \in \mathbb{N}$$

And $\{(F,A)_n\}$ = $\left\{\left(e_1, \left\{\frac{x_1}{(0.9)}, \frac{x_2}{(1/2n)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(1)}, \frac{x_2}{(1/3n)}\right\}\right)\right\} \forall\, n \in \mathbb{N}$

For the fuzzy soft sequential point $\{(P^n_{e_ix_j}, r)_E\}$, there exist fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that

$\{(P^n_{e_i x_j}, r)_E\} q_w \{(F,A)_n\}$ and $\{(P^n_{e_i x_j}, r)_E\}\ \overline{q}_w \{(G,A)_n\}$.

Similarly, for the fuzzy soft sequential point $\{(Q^n_{e_i x_j}, s)_E\}$, there exist fuzzy soft sequential open sets $\{(F,A)_n\}$ and $\{(G,A)_n\}$ such that $\{(Q^n_{e_i x_j}, s)_E\}\ q_w \{(G,A)_n\}$ and $\{(Q^n_{e_i x_j}, s)_E\} \overline{q}_w \{(F,A)_n\}$. It shows that the fuzzy soft sequential topological space is FSS - T_1 space.

Claim: $(U, \tilde{\tau}, E)$ is not FSS - T_2 space.

From the above part, we have $\{(P^n_{e_i x_j}, r)_E\}\ q_w\ \{(F,A)_n\}$, $\{(Q^n_{e_i x_j}, s)_E\}\ \ q_w\ \{(G,A)_n\}$.

It is enough to prove that $\{(P^n_{e_i x_j}, r)_E\}\ \overline{q}_w \overline{\{(G,A)_n\}}$, $\{(Q^n_{e_i x_j}, s)_E\}\ \ \overline{q}_w \overline{\{(F,A)_n\}}$.

Some of the fuzzy soft sequential closed sets are $0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(G,A)^c_n\}, \{(F,A)^c_n\}$.

$\overline{\{(F,A)_n\}} = 1^{\mathbb{N}}$ and $\overline{\{(G,A)_n\}} = \widetilde{\bigcap}\{1^{\mathbb{N}}, \{(G,A)^c_n\}\} = \{(G,A)^c_n\}$.

Therefore, there exists $e_1 \in A, x_1 \in U$ such that $P_{\mathrm{n}}(e_1)(x_1) + G^c_n(e_1)(x_1) \gtrsim 1$ for some $n \in \mathbb{N}$ implies $\{(P^n_{e_i x_j}, r)_E\}\ q_w\ \overline{\{(G,A)_n\}}$. Also, $\{(Q^n_{e_i x_j}, s)_E\} \overline{q}_w \overline{\{(F,A)_n\}}$. Hence $(U, \tilde{\tau}, E)$ is not FSS - T_2 space.

Theorem 3.3.6 A fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$ is a fuzzy soft sequential Hausdorff space. Then for any fuzzy soft sequential point $\{(P^n_{e_i x_j}, r)_E\}$ over U,

$\{(P^n_{e_i x_j}, r)_E\} = \widetilde{\bigcap}\{\overline{\{(F,A)_n\}};\ \{(F,A)_n\}$ is a FSS- weak Q-neighborhood of $\{(P^n_{e_i x_j}, r)_E\}\}$.

Proof. Let $(U, \tilde{\tau}, E)$ be a FSS - T_2 space and $\{(P^n_{e_i x_j}, r)_E\}$, $\{(Q^n_{e_i x_j}, s)_E\}$ be the distinct fuzzy soft sequential points. Then there exists a fuzzy soft sequential open set $\{(F,A)_n\}$ such that $\{(P^n_{e_i x_j}, r)_E\} q_w \{(F,A)_n\}$ and $\{(Q^n_{e_i x_j}, s)_E\}\ \overline{q}_w \overline{\{(F,A)_n\}}$. Suppose $\{(Q^n_{e_i x_j}, s)_E\} q_w \widetilde{\bigcap}\{\overline{\{(F,A)_n\}};\ \{(F,A)_n\}$ is a FSS- weak Q-neighborhood of $\{(P^n_{e_i x_j}, r)_E\}\}$. Hence $\{(Q^n_{e_i x_j}, s)_E\} q_w \{\overline{\{(F,A)_n\}}$, which is a contradiction. Thus, there is no fuzzy soft sequential point exists which is fuzzy soft sequential weakly quasi-coincident with $\overline{\{(F,A)_n\}}$.

4. Conclusion

In this paper, we studied separation axioms in fuzzy soft sequential topology. We introduced fuzzy soft sequential T_i axioms $(i = 0,1,2)$ using the concept of fuzzy soft sequential quasi-coincident relation via fuzzy soft sequential set and fuzzy soft sequential point and investigated some properties of these spaces with proper examples.

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