SOME NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY AND INDEX FOR REGULAR GRAPH

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Abstract: A regular graph is a graph in which all vertices have the same degree. If every vertex in a graph has degree k, then the graph is called a k-regular graph. In this paper, we introduce some open neighbourhood degree-based reachability matrices of graph and obtain its Energy and Estrada index. Also, we establish the bounds for this energy and index.

Keywords: Regular graph, Reachability matrix, Spectrum of a graph, Energy, Estrada Index.

1. INTRODUCTION

The **Energy** of a simple graph was introduced by Ivan Gutman in 1978 [14,15,19].

The Energy of a graph G, denoted by $E(G)$, is defined to be the sum of the absolute value of the eigenvalues of its adjacency matrix (i.e) $E(G) = \sum_{i=1}^{p} |\lambda_i|$. Various other energy measures based on different matrices have been discussed [5,17].

De la Pe˜na et.al., introduced the **Estrada index** of a graph in 2007[6].

Estrada index of the graph G is defined by $EE(G) = \sum_{i=1}^{p} e^{\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p$ are the eigenvalues of the adjacency matrix $A(G)$ of G [7,8,9,10,11,12,13,16].

Ramane et.al., introduced the degree sum matrix in 2013 [18]. Bala et. al., introduced the Reachability degree sum matrix in 2024 [1,3]. Bala et. al., introduced the Reachability energy of a graph in 2025 [2]. Bala et. al., introduced the Reachability average LH matrix and Inverse reachability degree sum matrix in 2025 [4].

2. PRELIMINARIES

In this section, we present the definitions and notations relevant to this work. Also, we introduce some matrices and present several lemmas based on these matrices, which form the foundation for the results discussed in the following sections.

Let G be a finite, simple, undirected graph with p vertices. Let $M_a(G)$ denote the matrix of a graph G. Since the matrix $M_a(G)$ under consideration are real and symmetric, its **eigenvalues** are real numbers and is denoted by ${\mu_1}^{(a)}, {\mu_2}^{(a)}, \ldots, {\mu_p}^{(a)}$, we label them in non-increasing order ${\mu_1}^{(a)} \geq {\mu_2}^{(a)} \geq \cdots \geq {\mu_p}^{(a)}$. (i.e.,) For $1 \leq i \leq p$, ${\mu_i}^{(a)}$ be the eigenvalues for $M_a(G)$. Also, we denote $R_a = \sum_{1 \leq i < j \leq p} (b_{ij})^2$ for $M_a(G)$. Now we denote $E_a(G)$ *and* $EE_a(G)$ be Energy and Estrada index of a graph based on a matrix $M_a(G)$.

The **Characteristic polynomial** of the matrix $M_a(G)$ is defined as $\phi(G,\mu) = \det(M_a(G) - \mu I)$, where I is the identity matrix and μ is a scalar. The roots of the equation $\phi(G,\mu) = 0$ is called the **eigenvalues** of the $M_a(G)$.

The collection of all eigenvalues is called **Spectrum** of a graph G[16]. If ${\mu_1}^{(a)} \geq {\mu_2}^{(a)} \geq \cdots \geq {\mu_p}^{(a)}$ are the distinct eigenvalues of G with multiplicities $m_1, m_2, \ldots, m_p$ respectively then

$$Spec\ (G) = \begin{pmatrix} {\mu_1}^{(a)} & {\mu_2}^{(a)} & \ldots & {\mu_p}^{(a)} \\ m_1 & m_2 & \ldots & m_p \end{pmatrix}.$$

The **Largest eigenvalue** ${\mu_1}^{(a)}$ of a connected graph G satisfies the inequality ${\mu_1}^{(a)} \leq \frac{2q}{p} \leq 1$.

${\mu_1}^{(a)}$is always positive.

The **Trace** of a square matrix $M_a(G)$, denoted by $tr(M_a(G))$ is the sum of its eigenvalues. (i.e.,) $tr\ (M_a(G)^k) = \sum_{i=1}^{p} \left({\mu_i}^{(a)}\right)^k$.

The **Determinant** of a square matrix $M_a(G)$, denoted by $Det\ \left(M_a(G)\right)$ is the product of its eigenvalues. (i.e.,) $Det\ (M_a(G)) = \prod_{i=1}^{p} {\mu_i}^{(a)}$.

The **Diameter** of a graph is the maximum distance over all pairs of vertices in G.

The **Reachability matrix [5]** $M_1(G)$ is defined as

$$M_1(G) \ = (b_{ij}) = \begin{cases} 1, & t_j\ is\ reachable\ from\ t_i \\ 0, & otherwise \end{cases}.$$

The **Reachability Energy [2]** $E_1(G)$ is defined by $E_1(G) \ = \sum_{i=1}^{p} \left|{\mu_i}^{(1)}\right|$.

The **Reachability Estrada index [2]** $EE_1(G)$ is defined by $EE_1(G) \ = \sum_{i=1}^{p} e^{{\mu_i}^{(1)}}$, where ${\mu_1}^{(1)} \geq {\mu_2}^{(1)} \geq \cdots \geq {\mu_p}^{(1)}$ are the eigenvalues of $M_1(G)$.

The **Reachability degree sum matrix [1]** $M_2(G)$ is defined by

$$M_2(G) = \left(b_{ij}\right) = \begin{cases} r_{ij} + d_i + d_j, & t_j\ is\ reachable\ from\ t_i \\ 0\ , & otherwise \end{cases}$$

where $d_i = degree\ of\ the\ vertex\ t_i$ and $r_{ij} = \begin{cases} 1\ , i \neq j \\ 0\ , i = j \end{cases}$.

The **Reachability degree sum Energy [1]** $E_2(G)$ is defined by $E_2(G) = \sum_{i=1}^{p} \left|{\mu_i}^{(2)}\right|$.

The **Reachability degree sum Estrada index [1]** $EE_2(G)$ is defined by

$EE_2(G) = \sum_{i=1}^{p} e^{{\mu_i}^{(2)}}$, where ${\mu_1}^{(2)} \geq {\mu_2}^{(2)} \geq \cdots \geq {\mu_p}^{(2)}$ are the eigenvalues of $M_2(G)$.

The **Reachability Average LH matrix [4]** $M_3(G)$ is defined as

$$M_3(G) = (b_{ij}) = \begin{cases} \frac{1}{2}\left(lcm(d_i, d_j) + hcf(d_i, d_j)\right), & t_j \text{ is reachable from } t_i \\ 0 & , \quad \text{otherwise} \end{cases}.$$

where $d_i = degree\ of\ the\ vertex\ t_i$.

The **Reachability Average LH Energy [4]** $E_3(G)$ is defined by $E_3(G) = \sum_{i=1}^{p} \left|\mu_i^{(3)}\right|$.

The **Reachability Average LH Estrada index [4]** $EE_3(G)$ is defined by

$EE_3(G) = \sum_{i=1}^{p} e^{\mu_i^{(3)}}$, where $\mu_1^{(3)} \geq \mu_2^{(3)} \geq \cdots \geq \mu_p^{(3)}$ are the eigenvalues of $M_3(G)$.

The **Inverse Reachability degree sum matrix [4]** $M_4(G)$ is defined as

$$M_4(G) = (b_{ij}) = \begin{cases} \frac{1}{r_{ij}+d_i+d_j}, & t_j \text{ is reachable from } t_i \\ 0 & , \quad \text{otherwise} \end{cases}.$$

where $d_i = degree\ of\ the\ vertex\ t_i$ and $r_{ij} = \begin{cases} 1\ , i \neq j \\ 0\ , i = j \end{cases}$.

The **Inverse Reachability degree sum Energy [4]** $E_4(G)$ is defined by $E_4(G) = \sum_{i=1}^{p} \left|\mu_i^{(4)}\right|$.

The **Inverse Reachability degree sum Estrada index [4]** $EE_4(G)$ is defined by

$EE_4(G) = \sum_{i=1}^{p} e^{\mu_i^{(4)}}$, where $\mu_1^{(4)} \geq \mu_2^{(4)} \geq \cdots \geq \mu_p^{(4)}$ are the eigenvalues of $M_4(G)$.

2.1 PROPOSED DEFINITIONS

In this subsection, we propose some open neighbourhood degree-based reachability matrices and their corresponding Energy and Estrada index for a graph.

For a vertex $t \in T(G)$ in a graph $G = (T, W)$, the **Open Neighbourhood** of t is the set of all vertices adjacent to t. The **Open Neighbourhood Degree** of t is defined as the sum of the degrees of all vertices in the open neighbourhood of t. It is denoted by $N_o(t)$.

Let $G(T, W)$ be a simple connected graph with p vertices and q edges. The **Open Neighbourhood Reachability sum matrix** $M_5(G)$ is the $p \times p$ matrix with

$$M_5(G) = (b_{ij}) = \begin{cases} N_o(t_i) + N_o(t_j), & t_j \text{ is reachable from } t_i \\ 0 & , \quad \text{otherwise} \end{cases}.$$

The sum of the absolute values of the eigenvalues of $M_5(G)$ is known as **Open Neighbourhood Reachability sum energy** $E_5(G)$ is defined by

$$E_5(G) = \sum_{i=1}^{p} \left|\mu_i^{(5)}\right|$$

If G is a connected graph with p vertices, then the **Open Neighbourhood Reachability sum Estrada index** $EE_5(G)$ is defined by

$$EE_5(G) = \sum_{i=1}^{p} e^{\mu_i^{(5)}}$$

where $\mu_1^{(5)} \geq \mu_2^{(5)} \geq \cdots \geq \mu_p^{(5)}$ are the eigenvalues of $M_5(G)$.

Let $G(T, W)$ be a simple connected graph with p vertices and q edges. The **Open Neighbourhood Reachability degree sum matrix** $M_6(G)$ is the $p \times p$ matrix with

$$M_6(G) = (b_{ij}) = \begin{cases} r_{ij} + N_o(t_i) + N_o(t_j), & t_j \text{ is reachable from } t_i \\ 0 \quad , & \text{otherwise} \end{cases}.$$

where $r_{ij} = \begin{cases} 1 \, , i \neq j \\ 0 \, , i = j \end{cases}$.

The sum of the absolute values of the eigenvalues of $M_6(G)$ is known as **Open Neighbourhood Reachability degree sum energy** $E_6(G)$ is defined by

$$E_6(G) = \sum_{i=1}^{p} \left|\mu_i^{(6)}\right|$$

If G is a connected graph with p vertices, then the **Open Neighbourhood Reachability degree sum Estrada index** $EE_6(G)$ is defined by

$$EE_6(G) = \sum_{i=1}^{p} e^{\mu_i^{(6)}}$$

where $\mu_1^{(6)} \geq \mu_2^{(6)} \geq \cdots \geq \mu_p^{(6)}$ are the eigenvalues of $M_6(G)$.

Let $G(T, W)$ be a simple connected graph with p vertices and q edges. The **Inverse Open Neighbourhood Reachability sum matrix** $M_7(G)$ is the $p \times p$ matrix with

$$M_7(G) = (b_{ij}) = \begin{cases} \frac{1}{N_o(t_i) + N_o(t_j)}, & t_j \text{ is reachable from } t_i \\ 0 \quad , & \text{otherwise} \end{cases}.$$

The sum of the absolute values of the eigenvalues of $M_7(G)$ is known as **Inverse Open Neighbourhood Reachability sum energy** $E_7(G)$ is defined by

$$E_7(G) = \sum_{i=1}^{p} \left|\mu_i^{(7)}\right|$$

If G is a connected graph with p vertices, then the **Inverse Open Neighbourhood Reachability sum Estrada index** $EE_7(G)$ is defined by

$$EE_7(G) = \sum_{i=1}^{p} e^{\mu_i^{(7)}}$$

where $\mu_1^{(7)} \geq \mu_2^{(7)} \geq \cdots \geq \mu_p^{(7)}$ are the eigenvalues of $M_7(G)$.

Let $G(T,W)$ be a simple connected graph with p vertices and q edges. The **Inverse Open Neighbourhood Reachability degree sum matrix** $M_8(G)$ is the $p \times p$ matrix with

$$M_8(G) = (b_{ij}) = \begin{cases} \frac{1}{r_{ij}+N_o(t_i)+N_o(t_j)}, & t_j \text{ is reachable from } t_i \\ 0, & \text{otherwise} \end{cases}.$$

where $r_{ij} = \begin{cases} 1, i \neq j \\ 0, i = j \end{cases}$.

The sum of the absolute values of the eigenvalues of $M_8(G)$ is known as **Inverse Open Neighbourhood Reachability degree sum energy** $E_8(G)$ is defined by

$$E_8(G) = \sum_{i=1}^{p} \left|\mu_i^{(8)}\right|$$

If G is a connected graph with p vertices, then the **Inverse Open Neighbourhood Reachability degree sum Estrada index** $EE_8(G)$ is defined by

$$EE_8(G) = \sum_{i=1}^{p} e^{\mu_i^{(8)}}$$

where $\mu_1^{(8)} \geq \mu_2^{(8)} \geq \cdots \geq \mu_p^{(8)}$ are the eigenvalues of $M_8(G)$.

Let $G(T,W)$ be a simple connected graph with p vertices and q edges. The **Open Neighbourhood Reachability average LH matrix** $M_9(G)$ is the $p \times p$ matrix with

$$M_9(G) = (b_{ij}) = \begin{cases} \frac{1}{2}\left(lcm\left(N_o(t_i), N_o(t_j)\right) + hcf(N_o(t_i), N_o(t_j))\right), & t_j \text{ is reachable from } t_i \\ 0, & \text{otherwise} \end{cases}$$

The sum of the absolute values of the eigenvalues of $M_9(G)$ is known as **Open Neighbourhood Reachability average LH energy** $E_9(G)$ is defined by

$$E_9(G) = \sum_{i=1}^{p} \left|\mu_i^{(9)}\right|$$

If G is a connected graph with p vertices, then the **Open Neighbourhood Reachability average LH Estrada index** $EE_9(G)$ is defined by

$$EE_9(G) = \sum_{i=1}^{p} e^{\mu_i^{(9)}}$$

where $\mu_1^{(9)} \geq \mu_2^{(9)} \geq \cdots \geq \mu_p^{(9)}$ are the eigenvalues of $M_9(G)$.

2.2 SUPPORTING LEMMAS

In this subsection, we discuss some lemmas and using that we will calculate the bounds for largest eigenvalue for our proposed open neighbourhood degree-based reachability matrices.

Lemma 2.2.1:

Let G be a connected graph of order p. For $5 \leq a \leq 9$ *and* $1 \leq i \leq p$, ${\mu_i}^{(a)}$ be the eigenvalues for M_a then

$tr\left(M_a(G)\right) = \sum_{i=1}^{p} {\mu_i}^{(a)} = 0$ and

$tr\,(M_a(G)^2) = \sum_{i=1}^{p} ({\mu_i}^{(a)})^2 = 2\,R_a, where\ R_a = \sum_{1\leq i<j\leq p}\left(b_{ij}\right)^2$

Proof:

By usual notation, we have $\sum_{i=1}^{p} {\mu_i}^{(a)}$ is equal to the trace of a matrix.

Now, $\sum_{i=1}^{p} {\mu_i}^{(a)} = trace\left(M_a(G)\right) = \sum_{i=j=1}^{p} R_a = 0.$

Moreover, for $i = 1,2,\dots,p,$ the $(i,i)th$ entry of $(M_a(G))^2$ is equal to $\sum_{j=1}^{p}\left(b_{ij}\right)\left(b_{ji}\right) = \sum_{i=1}^{p}(b_{ij})^2$

$$\sum_{i=1}^{p}({\mu_i}^{(a)})^2 = trace\,(M_a(G))^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}(b_{ij})^2 = 2\,R_a.$$

Hence the result.

Lemma 2.2.2:

Let G represents a graph with p vertices which satisfies the inequality for $5 \leq a \leq 9$, $|Det\,(M_a(G))| \leq (2\,R_a)^{\frac{p}{2}}$.

Proof:

We know that, $|Det\,(M_a(G))| = \prod_{i=1}^{p}\left|({\mu_i}^{(a)})\right|$

$$= \left|({\mu_1}^{(a)})\right|\left|({\mu_2}^{(a)})\right|\dots\left|({\mu_p}^{(a)})\right| \leq \left|({\mu_1}^{(a)})\right|\left|({\mu_1}^{(a)})\right|\dots\left|({\mu_1}^{(a)})\right| \leq \left|({\mu_1}^{(a)})\right|^p$$

$$\leq \left(\sqrt{2\sum_{1\leq i<j\leq p}\left(b_{ij}\right)^2}\right)^p \leq (2\,R_a)^{\frac{p}{2}}$$

$$\therefore |Det\,(M_a(G))| \leq (2\,R_a)^{\frac{p}{2}}.$$

Lemma 2.2.3:

If G is any graph with p vertices then ${\mu_1}^{(a)} \leq \sqrt{\frac{2\,(p-1)R_a}{p}}$ for $5 \leq a \leq 9$.

Proof:

Using *Cauchy Schwartz inequality*, by setting $x_i = 1$ and $y_i = \mu_i{}^{(a)}$ for $i = 2,3,\dots,p$ then we get,

$$\left(\sum_{i=2}^{p}\left|\mu_i{}^{(a)}\right|\right)^2 \leq (p-1)\left(\sum_{i=1}^{p}\mu_i{}^{(a)2}\right)$$

$$\left(-\mu_1{}^{(a)}\right)^2 \leq (p-1)\left(2\sum_{1\leq i<j\leq p}\left(b_{ij}\right)^2 - \mu_1{}^{(a)2}\right)$$

$$\Rightarrow \mu_1{}^{(a)} \leq \sqrt{\frac{2\,(p-1)R_a}{p}}.$$

Lemma 2.2.4:

Consider a connected graph G with p vertices. The largest eigenvalue $\mu_1{}^{(a)}$ of G satisfies the inequality $\left|\mu_1{}^{(a)}\right| \geq |Det\ (M_a(G))|^{\frac{1}{p}}\ for\ 5 \leq a \leq 9.$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $\left|\mu_1{}^{(a)}\right|, \left|\mu_2{}^{(a)}\right|, \dots, \left|\mu_p{}^{(a)}\right|$, we get,

$$\frac{\left|\mu_1{}^{(a)} + \mu_2{}^{(a)} + \cdots + \mu_p{}^{(a)}\right|}{p} \geq \left|\mu_1{}^{(a)}\mu_2{}^{(a)} \dots \mu_p{}^{(a)}\right|^{\frac{1}{p}}$$

$$\Rightarrow \frac{\left|\mu_1{}^{(a)}\right| + \left|\mu_1{}^{(a)}\right| + \cdots + \left|\mu_1{}^{(a)}\right|}{p} \geq |Det\ M_a(G)|^{\frac{1}{p}}$$

$$\left|\mu_1{}^{(a)}\right| \geq |Det\ (M_a(G))|^{\frac{1}{p}}.$$

Lemma 2.2.5:

Let G be a connected graph with p vertices, the largest eigenvalue $\mu_1{}^{(a)}$ of G satisfies the inequality $\left|\mu_1{}^{(a)}\right| \geq \frac{|Det\ (M_a(G))|^{\frac{1}{p}}}{\sqrt{p}}\ for\ 5 \leq a \leq 9.$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $\left|\mu_1{}^{(a)}\right|, \left|\mu_2{}^{(a)}\right|, \dots, \left|\mu_p{}^{(a)}\right|$, we get,

$$\frac{\left|\mu_1{}^{(a)} + \mu_2{}^{(a)} + \cdots + \mu_p{}^{(a)}\right|}{p} \geq \left|\mu_1{}^{(a)}\mu_2{}^{(a)} \dots \mu_p{}^{(a)}\right|^{\frac{1}{p}}$$

$$\frac{|\mu_1{}^{(a)}| + |\mu_1{}^{(a)}| + \cdots + |\mu_1{}^{(a)}|}{\sqrt{p}} \geq \frac{|\mu_1{}^{(a)}| + |\mu_1{}^{(a)}| + \cdots + |\mu_1{}^{(a)}|}{p} \geq |\mu_1{}^{(a)}\mu_2{}^{(a)} \ldots \mu_p{}^{(a)}|^{\frac{1}{p}}$$

$$\Rightarrow |\mu_1{}^{(a)}| \geq \frac{|Det\ (M_a(G))|^{\frac{1}{p}}}{\sqrt{p}}.$$

3. OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY AND ITS BOUNDS

In this section, we obtain some open neighbourhood degree-based reachability energy of regular graph and calculate its bounds.

3.1. OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY OF REGULAR GRAPH

In this subsection, we find some open neighbourhood degree-based reachability energies $E_a(G), 5 \leq a \leq 9$ for a regular graph.

Theorem 3.1.1:

Let G be a k- regular graph of order p then

(i) $E_5(G) = 4k^2(p-1)$

(ii) $E_6(G) = 2\ (2k^2+1)(p-1)$

(iii) $E_7(G) = \frac{p-1}{k^2}$

(iv) $E_8(G) = \frac{2(p-1)}{2k^2+1}$

(v) $E_9(G) = 2k^2(p-1)$

Proof:

Consider k- Regular graph G with p vertices.

(i) The open neighbourhood reachability sum matrix $\mathrm{M}_5(G\ )$ is

$$\mathrm{M}_5(G) = \begin{pmatrix} 0 & 2k^2 & 2k^2 & \cdots & 2k^2 & 2k^2 \\ 2k^2 & 0 & 2k^2 & \cdots & 2k^2 & 2k^2 \\ 2k^2 & 2k^2 & 0 & \cdots & 2k^2 & 2k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2k^2 & 2k^2 & 2k^2 & \cdots & 0 & 2k^2 \\ 2k^2 & 2k^2 & 2k^2 & \cdots & 2k^2 & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_5(G)$ using the relation,

$$\phi(G,\mu) = \det(\mathrm{M}_5(G) - \mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & 2k^2 & 2k^2 & \cdots & 2k^2 & 2k^2 \\ 2k^2 & -\mu & 2k^2 & \cdots & 2k^2 & 2k^2 \\ 2k^2 & 2k^2 & -\mu & \cdots & 2k^2 & 2k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2k^2 & 2k^2 & 2k^2 & \cdots & -\mu & 2k^2 \\ 2k^2 & 2k^2 & 2k^2 & \cdots & 2k^2 & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_5(G)$ is

$$\begin{pmatrix} -2k^2 & 2k^2(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The open neighbourhood reachability sum energy $\mathrm{E}_5(G)$ can be determined as follows:

$$\mathrm{E}_5(G)=\sum_{i=1}^{p}\left|\mu_{\mathrm{i}}{}^{(5)}\right|$$

$$=(|-2k^2|\times(p-1))+(|2k^2(p-1)|\times 1)$$

$$\mathrm{E}_5(G)=4k^2(p-1).$$

(ii) The open neighbourhood reachability degree sum matrix $\mathrm{M}_6(G\ )$ is

$$\mathrm{M}_6(G)=\begin{pmatrix} 0 & 2k^2+1 & 2k^2+1 & \cdots & 2k^2+1 & 2k^2+1 \\ 2k^2+1 & 0 & 2k^2+1 & \cdots & 2k^2+1 & 2k^2+1 \\ 2k^2+1 & 2k^2+1 & 0 & \cdots & 2k^2+1 & 2k^2+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2k^2+1 & 2k^2+1 & 2k^2+1 & \cdots & 0 & 2k^2+1 \\ 2k^2+1 & 2k^2+1 & 2k^2+1 & \cdots & 2k^2+1 & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_6(G)$ using the relation,

$$\phi(G,\mu)=\det(\mathrm{M}_6(G)-\mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & 2k^2+1 & 2k^2+1 & \cdots & 2k^2+1 & 2k^2+1 \\ 2k^2+1 & -\mu & 2k^2+1 & \cdots & 2k^2+1 & 2k^2+1 \\ 2k^2+1 & 2k^2+1 & -\mu & \cdots & 2k^2+1 & 2k^2+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2k^2+1 & 2k^2+1 & 2k^2+1 & \cdots & -\mu & 2k^2+1 \\ 2k^2+1 & 2k^2+1 & 2k^2+1 & \cdots & 2k^2+1 & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_6(G)$ is

$$\begin{pmatrix} -(2k^2+1) & (2k^2+1)(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The open neighbourhood reachability degree sum energy $\mathrm{E}_6(G)$ can be determined as follows:

$$\mathrm{E}_6(G)=\sum_{i=1}^{p}\left|\mu_{\mathrm{i}}{}^{(6)}\right|$$

$$=(|-(2k^2+1)|\times(p-1))+(|(2k^2+1)(p-1)|\times 1)$$

$$\mathrm{E}_6(G) = 2(2k^2+1)(p-1).$$

(iii) The inverse open neighbourhood reachability sum matrix $\mathrm{M}_7(G)$ is

$$\mathrm{M}_7(G) = \begin{pmatrix} 0 & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \frac{1}{2k^2} & 0 & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \frac{1}{2k^2} & \frac{1}{2k^2} & 0 & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2k^2} & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & 0 & \frac{1}{2k^2} \\ \frac{1}{2k^2} & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_7(G)$ using the relation,

$$\phi(G,\mu) = \det(\mathrm{M}_7(G) - \mu I), \textit{where I is the idendity matrix.}$$

$$\phi(G,\mu) = \begin{vmatrix} -\mu & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \frac{1}{2k^2} & -\mu & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \frac{1}{2k^2} & \frac{1}{2k^2} & -\mu & \cdots & \frac{1}{2k^2} & \frac{1}{2k^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2k^2} & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & -\mu & \frac{1}{2k^2} \\ \frac{1}{2k^2} & \frac{1}{2k^2} & \frac{1}{2k^2} & \cdots & \frac{1}{2k^2} & -\mu \end{vmatrix} = 0$$

Hence, the spectrum of $\mathrm{M}_7(G)$ is

$$\begin{pmatrix} \frac{-1}{2k^2} & \frac{p-1}{2k^2} \\ p-1 & 1 \end{pmatrix}.$$

The inverse open neighbourhood reachability sum energy $\mathrm{E}_7(G)$ can be determined as follows:

$$\mathrm{E}_7(G) = \sum_{i=1}^{p} \left|\mu_{\mathrm{i}}^{(7)}\right|$$

$$= \left(\left|\frac{-1}{2k^2}\right| \times (p-1)\right) + \left(\left|\frac{p-1}{2k^2}\right| \times 1\right)$$

$$\mathrm{E}_7(G) = \frac{p-1}{k^2}.$$

(iv) The inverse open neighbourhood reachability degree sum matrix $\mathrm{M}_8(G)$ is

$$\mathrm{M}_8(G)=\begin{pmatrix} 0 & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & 0 & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & 0 & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & 0 & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_8(G)$ using the relation,

$$\phi(G,\mu)=\det(\mathrm{M}_8(G)-\mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu)=\begin{vmatrix} -\mu & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & -\mu & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & -\mu & \cdots & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & -\mu & \frac{1}{2k^2+1} \\ \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \frac{1}{2k^2+1} & \cdots & \frac{1}{2k^2+1} & -\mu \end{vmatrix}=0$$

Hence, the spectrum of $\mathrm{M}_8(G)$ is

$$\begin{pmatrix} \frac{-1}{2k^2+1} & \frac{p-1}{2k^2+1} \\ p-1 & 1 \end{pmatrix}.$$

The inverse open neighbourhood reachability degree sum energy $\mathrm{E}_8(G)$ can be determined as follows:

$$\mathrm{E}_8(G)=\sum_{i=1}^{p}\left|\mu_{\mathrm{i}}{}^{(8)}\right|$$

$$=\left(\left|\frac{-1}{2k^2+1}\right|\times(p-1)\right)+\left(\left|\frac{p-1}{2k^2+1}\right|\times 1\right)$$

$$\mathrm{E}_8(G)=\frac{2(p-1)}{2k^2+1}.$$

(v) The open neighbourhood reachability average LH matrix $\mathrm{M}_9(G\ )$ is

$$\mathrm{M}_9(G) = \begin{pmatrix} 0 & k^2 & k^2 & \cdots & k^2 & k^2 \\ k^2 & 0 & k^2 & \cdots & k^2 & k^2 \\ k^2 & k^2 & 0 & \cdots & k^2 & k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k^2 & k^2 & k^2 & \cdots & 0 & k^2 \\ k^2 & k^2 & k^2 & \cdots & k^2 & 0 \end{pmatrix}$$

Let us find the spectrum of $\mathrm{M}_9(G)$ using the relation,

$$\phi(G,\mu) = \det(\mathrm{M}_9(G) - \mu I)\,, where\ I\ is\ the\ idendity\ matrix.$$

$$\phi(G,\mu) = \begin{vmatrix} -\mu & k^2 & k^2 & \cdots & k^2 & k^2 \\ k^2 & -\mu & k^2 & \cdots & k^2 & k^2 \\ k^2 & k^2 & -\mu & \cdots & k^2 & k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k^2 & k^2 & k^2 & \cdots & -\mu & k^2 \\ k^2 & k^2 & k^2 & \cdots & k^2 & -\mu \end{vmatrix} = 0$$

Hence, the spectrum of $\mathrm{M}_9(G)$ is

$$\begin{pmatrix} -k^2 & k^2(p-1) \\ p-1 & 1 \end{pmatrix}.$$

The open neighbourhood reachability average LH energy $\mathrm{E}_9(G)$ can be determined as follows:

$$\mathrm{E}_9(G) = \sum_{i=1}^{p} \left|\mu_{\mathrm{i}}{}^{(9)}\right|$$

$$= (|-(k^2)| \times (p-1)) + (|k^2(p-1)| \times 1)$$

$$\mathrm{E}_9(G) = 2k^2(p-1).$$

3.2 BOUNDS FOR OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ENERGY OF GRAPH

In this subsection, we obtain bounds for open neighbourhood degree-based reachability energy of regular graph.

Theorem 3.2.1:

If G be a connected graph, then $\sqrt{2R_a} \leq E_a(G) \leq \sqrt{2pR_a}$ for $5 \leq a \leq 9$.

Proof:

By *Cauchy-Schwartz inequality*,

$$\left(\sum_{i=1}^{p} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{p} x_i{}^2\right)\left(\sum_{i=1}^{p} y_i{}^2\right)$$

Consider, $x_i = 1$ and $y_i = \left|\mu_i^{(a)}\right|$, then

$$\left(\sum_{i=1}^{p}\left|\mu_i^{(a)}\right|\right)^2 \leq p\left(\sum_{i=1}^{p}(\mu_i^{(a)})^2\right)$$

$$E_a(G)^2 \leq 2pR_a$$

$$E_a(G) \leq \sqrt{2pR_a}.$$

which gives the required upper bound for $E_a(G)$.

Consider, $\left(E_a(G)\right)^2 = \left(\sum_{i=1}^{p}\left|\mu_i^{(a)}\right|\right)^2 \geq \sum_{i=1}^{p}\left|\mu_i^{(a)}\right|^2 = 2R_a$

$$E_a(G) \geq \sqrt{2R_a}.$$

which gives the required lower bound for $E_a(G)$.

$$\sqrt{2R_a} \leq E_a(G) \leq \sqrt{2pR_a}.$$

Hence the result.

Theorem 3.2.2:

Let G be a connected graph and let $|Det\,(M_a(G))|$ be the absolute value of the determinant of the $M_a(G)$ of a graph then for $5 \leq a \leq 9$,

$$\sqrt{2R_a + p|Det\,(M_a(G))|^{\frac{2}{p}}} \leq E_a(G) \leq \sqrt{2pR_a}.$$

Proof:

By the theorem 3.2.1, we have the upper bound for $E_a(G)$ as $E_a(G) \leq \sqrt{2pR_a}$.

Now, we obtain the lower bound for $E_a(G)$.

Consider,

$$\left(E_a(G)\right)^2 = \left(\sum_{i=1}^{p}\left|\mu_i^{(a)}\right|\right)^2 = \sum_{i=1}^{p}\left|\mu_i^{(a)}\right|^2 + 2\sum_{1\leq i<j\leq p}\left|\mu_i^{(a)}\right|\left|\mu_j^{(a)}\right|$$

$$= 2R_a + \sum_{i\neq j}\left|\mu_i^{(a)}\right|\left|\mu_j^{(a)}\right|$$

From *Arithmetic – Geometric mean inequality*, we have,

$$\frac{1}{p(p-1)}\sum_{i\neq j}\left|\mu_i^{(a)}\right|\left|\mu_j^{(a)}\right| \geq \left(\prod_{i\neq j}\left|\mu_i^{(a)}\right|\left|\mu_j^{(a)}\right|\right)^{\frac{1}{p(p-1)}}$$

$$= \left(\prod_{i=1}^{p} \left| \mu_i{}^{(a)} \right|^{2(p-1)} \right)^{\frac{1}{p(p-1)}} = |Det\ (M_a(G))|^{\frac{2}{p}}.$$

which implies that

$$\left(E_a(G)\right)^2 \geq 2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}$$

$$E_a(G) \geq \sqrt{2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}}\ .$$

which gives the required lower bound for $E_a(G)$.

$$\sqrt{2R_a + p|Det\ (M_a(G))|^{\frac{2}{p}}}\ \leq E_a(G) \leq \sqrt{2pR_a}.$$

Hence the result.

Theorem 3.2.3:

Let G be a connected graph with p vertices and $M_a(G)$ be a non-singular matrix then for $5 \leq a \leq 9$,

$$p|Det\ (M_a(G))|^{\frac{1}{p}} \leq E_a(G) \leq \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}.$$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $\left|\mu_1{}^{(a)}\right|, \left|\mu_2{}^{(a)}\right|, \dots, \left|\mu_p{}^{(a)}\right|$, we get,

$$\frac{\left|\mu_1{}^{(a)} + \mu_2{}^{(a)} + \cdots + \mu_p{}^{(a)}\right|}{p} \geq \left|\mu_1{}^{(a)}\mu_2{}^{(a)} \dots \mu_p{}^{(a)}\right|^{\frac{1}{p}}$$

$$\sum_{i=1}^{p} \left|\mu_i{}^{(a)}\right| \geq p|Det\ (M_a(G))|^{\frac{1}{p}}$$

$$E_a(G) \geq p|Det\ (M_a(G))|^{\frac{1}{p}}$$

which gives a lower bound for $E_a(G)$.

By using lemma 2.2.4, we have $\left|\mu_1{}^{(a)}\right| \geq |Det\ (M_a(G))|^{\frac{1}{p}}$

$$\left|\mu_1{}^{(a)}\right| \sum_{i=1}^{p} \left|\mu_i{}^{(a)}\right| \geq |Det\ (M_a(G))|^{\frac{1}{p}} \sum_{i=1}^{p} \left|\mu_i{}^{(a)}\right|$$

$$\Rightarrow p\left|\mu_1{}^{(a)}\right|^2 \geq |Det\ (M_a(G))|^{\frac{1}{p}} \left(E_a(G)\right)$$

$$E_a(G) \leq \frac{p\left|\mu_1{}^{(a)}\right|^2}{|Det\ (M_a(G))|^{\frac{1}{p}}}$$

Since $\left|\mu_1{}^{(a)}\right|^2 \leq 2R_a$, $E_a(G) \leq \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}$

which gives an upper bound for $E_a(G)$.

$$p|Det\ (M_a(G))|^{\frac{1}{p}} \leq E_a(G) \leq \frac{2pR_a}{|Det\ (M_a(G))|^{\frac{1}{p}}}.$$

Hence the theorem.

4. OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX AND ITS BOUNDS

In this section, we will introduce and obtain some open neighbourhood degree-based reachability Estrada index and its bounds. Additionally, we will determine an upper bound for open neighbourhood degree-based reachability Estrada index in terms of open neighbourhood degree-based reachability energy of graph.

4.1. OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX OF GRAPH

In this subsection, we introduce some open neighbourhood degree-based reachability Estrada index of a graph and derive bounds for it

Definition 4.1.1:

If G be a connected graph with p vertices, then the Estrada index $\mathrm{EE}_a(G)$ based on a matrix $M_a(G)$ for $5 \leq a \leq 9$ is defined by

$$EE_a(G) = \sum_{i=1}^{p} e^{\mu_i{}^{(a)}}$$

where $\mu_1{}^{(a)} \geq \mu_2{}^{(a)} \geq \cdots \geq \mu_p{}^{(a)}$ are the eigenvalues of $M_a(G)$.

Let $V_k = \sum_{i=1}^{p}(\mu_i{}^{(a)})^k$, where

$$V_0 = \sum_{i=1}^{p}(\mu_i{}^{(a)})^0 = p; V_1 = \sum_{i=1}^{p}(\mu_i{}^{(a)})^1 = 0; V_2 = \sum_{i=1}^{p}(\mu_i{}^{(a)})^2 = 2\sum_{1\leq i<j\leq p}\left(a_{ij}\right)^2 = 2R_a$$

Also, we have, $V_k = tr\ (M_a(G)^k)$.

Then, $EE_a(G) = \sum_{k=0}^{\infty}\frac{V_k}{k!}$.

4.2. BOUNDS FOR OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX OF GRAPH

In this subsection, we will obtain the upper bound and lower bound for open neighbourhood degree-based reachability Estrada index of graph.

Theorem 4.2.1:

Let G be a connected graph with diameter less than or equal to 2 then for $5 \leq a \leq 9$,

$$\sqrt{p^2 + 4\mathrm{R}_a} \leq \mathrm{EE}_a(G) \leq p - 1 + e^{\sqrt{2\mathrm{R}_a}}.$$

Proof:

From the definition 4.1.1, $EE_a(G) = \sum_{i=1}^{p} e^{\mu_i{}^{(a)}}$

$$EE_a{}^2(G) = \left(\sum_{i=1}^{p} e^{\mu_i{}^{(a)}}\right)^2 = \sum_{i=1}^{p} e^{2\mu_i{}^{(a)}} + 2\sum_{1\leq i<j\leq p} e^{\mu_i{}^{(a)}} e^{\mu_j{}^{(a)}}$$

Consider the 2nd term of the above equation, by using *Arithmetic-Geometric Mean Inequality*, we have

$$2\sum_{1\leq i<j\leq p} e^{\mu_i{}^{(a)}} e^{\mu_j{}^{(a)}} \geq p(p-1)\left(\prod_{1\leq i<j\leq p} e^{\mu_i{}^{(a)}} e^{\mu_j{}^{(a)}}\right)^{\frac{2}{p(p-1)}}$$

$$= p(p-1)\left(\left(\prod_{i=1}^{p} e^{\mu_i{}^{(a)}}\right)^{p-1}\right)^{\frac{2}{p(p-1)}}$$

$$= p(p-1)(e^{V_1})^{\frac{2}{p}} = p(p-1)$$

$$2\sum_{1\leq i<j\leq p} e^{\mu_i{}^{(a)}} e^{\mu_j{}^{(a)}} \geq p(p-1).$$

Consider,

$$\sum_{i=1}^{p} e^{2\mu_i{}^{(a)}} = \sum_{i=1}^{p}\sum_{k\geq 0} \frac{\left(2\mu_i{}^{(a)}\right)^k}{k!} = p + 4R_a + \sum_{i=1}^{p}\sum_{k\geq 3} \frac{\left(2\mu_i{}^{(a)}\right)^k}{k!}$$

Since we require lower bound as good as possible, it holds reasonable to replace $\sum_{k\geq 3} \frac{\left(2\mu_i{}^{(a)}\right)^k}{k!}$ by $4\frac{\left(\mu_i{}^{(a)}\right)^k}{k!}$. Further, we use a multiplier $t \in [0,4]$ instead of 4. We get,

$$\sum_{i=1}^{p} e^{2\mu_i{}^{(a)}} \geq p + 4R_a + t\sum_{i=1}^{p}\sum_{k\geq 3} \frac{\left(\mu_i{}^{(a)}\right)^k}{k!}$$

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} \geq p + 4R_a - tp - tR_a + t\sum_{i=1}^{p}\sum_{k\geq 0}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$\sum_{i=1}^{p} e^{2\mu_i^{(a)}} \geq p(1-t) + (4-t)R_a + t.EE_a(G)$$

Then solving for $\mathrm{EE}_a(G)$,

$$EE_a{}^2(G) \geq p(1-t) + (4-t)R_a + t.EE_a(G) + p(p-1)$$

$$EE_a{}^2(G) \geq p^2 + 4R_a + t\,[EE_a(G) - R_a - p]$$

For $p \geq 2$, the best lower bound for $\mathrm{EE}_a(G)$ is attained when $t = 0$.

$$EE_a{}^2(G) \geq p^2 + 4R_a$$

$$EE_a(G) \geq \sqrt{p^2 + 4R_a}.$$

which gives the required lower bound for $\mathrm{EE}_a(G)$.

From the definition 4.1.1,

$$EE_a(G) = \sum_{k=0}^{\infty}\frac{V_k}{k!} = p + \sum_{k=1}^{\infty}\frac{V_k}{k!}$$

$$EE_a(G) \leq p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$EE_a(G) \leq p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$\leq p + \sum_{k\geq 1}\frac{1}{k!}\sum_{i=1}^{p}\left(\left(\mu_i^{(a)}\right)^2\right)^{\frac{k}{2}} = p + \sum_{k\geq 1}\frac{1}{k!}(2R_a)^{\frac{k}{2}}$$

$$= p - 1 + \sum_{k\geq 0}\frac{\left(\sqrt{2R_a}\right)^k}{k!} = p - 1 + e^{\sqrt{2R_a}}$$

$$EE_a(G) \leq p - 1 + e^{\sqrt{2R_a}}.$$

which gives required upper bound for $\mathrm{EE}_a(G)$.

$$\therefore \sqrt{p^2 + 4\mathrm{R}_a} \leq \mathrm{EE}_a(G) \leq p - 1 + e^{\sqrt{2\mathrm{R}_a}}.$$

Hence the theorem.

Theorem 4.2.2:

Let G be a connected graph of order p then for $5 \leq a \leq 9$,

$$\frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1} \leq EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p(p-1).$$

Proof:

By known result, let x_i, $1 \leq i \leq p$ be any real numbers, then

$$p\left[\frac{1}{p}\sum_{i=1}^{p} x_i - \left(\prod_{i=1}^{p} x_i\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} x_i - \left(\sum_{i=1}^{p}\sqrt{x_i}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} x_i - \left(\prod_{i=1}^{p} x_i\right)^{\frac{1}{p}}\right]$$

By setting $x_i = e^{\mu_i^{(a)}}$ for $i = 1,2,\dots,p$, we have

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{\mu_i^{(a)}}}\right)^2$$

$$\leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{\frac{1}{p}}\right]$$

Consider,

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{\mu_i^{(a)}}}\right)^2$$

$$\sum_{i=1}^{p} e^{\mu_i^{(a)}} - p\left(e^{\sum_{i=1}^{p}\mu_i^{(a)}}\right)^{\frac{1}{p}} \leq p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{\mu_i^{(a)}}}\right)^2$$

$$\Rightarrow \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p \leq (p-1)\sum_{i=1}^{p} e^{\mu_i^{(a)}}$$

$$\sum_{i=1}^{p} e^{\mu_i^{(a)}} \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1}$$

$$EE_a(G) \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1}.$$

which gives the lower bound for $\mathrm{EE}_a(G)$.

Consider,

$$p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{\mu_i^{(a)}}}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\prod_{i=1}^{p} e^{\mu_i^{(a)}}\right)^{\frac{1}{p}}\right]$$

$$p\sum_{i=1}^{p} e^{\mu_i^{(a)}} - \left(\sum_{i=1}^{p}\sqrt{e^{\mu_i^{(a)}}}\right)^2 \leq (p-1)\sum_{i=1}^{p} e^{\mu_i^{(a)}} - p(p-1)\left(e^{\sum_{i=1}^{p}\mu_i^{(a)}}\right)^{\frac{1}{p}}$$

$$\Longrightarrow EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p(p-1).$$

which gives an upper bound for $\mathrm{EE}_a(G)$.

$$\frac{\left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p}{p-1} \leq EE_a(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\mu_i^{(a)}}{2}}\right)^2 - p(p-1).$$

Hence the theorem.

4.3. UPPER BOUND FOR THE OPEN NEIGHBOURHOOD DEGREE BASED REACHABILITY ESTRADA INDEX IN TERMS OF THE CORRESPONDING ENERGY

In this subsection, we find two upper bounds for some open neighbourhood degree-based reachability Estrada index based on the corresponding open neighbourhood degree-based reachability Energy. We consider a connected graphs with diameter at most 2.

Theorem 4.2.1:

Let G be a connected graph of diameter not greater than 2 then for $5 \leq a \leq 9$,

$$EE_a(G) - E_a(G) \leq p - 1 - \sqrt{2R_a} + e^{\sqrt{2R_a}}$$

and

$$EE_a(G) \leq p - 1 + e^{E_a(G)}.$$

Proof:

From definition 4.1.1, we have,

$$EE_a(G) = p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left(\mu_i^{(a)}\right)^k}{k!}$$

$$\leq p + \sum_{i=1}^{p}\sum_{k\geq 1}\frac{\left|\mu_i^{(a)}\right|^k}{k!}$$

$$EE_a(G) \leq p + E_a(G) + \sum_{i=1}^{p} \sum_{k\geq 2} \frac{\left|\mu_i{}^{(a)}\right|^k}{k!}$$

$$EE_a(G) - E_a(G) \leq p - 1 - \sqrt{2R_a} + e^{\sqrt{2R_a}}$$

Another approximation to connect $\mathrm{EE}_a(G)$ and $\mathrm{E}_a(G)$can be seen as follows:

$$EE_a(G) \leq p + \sum_{i=1}^{p} \sum_{k\geq 1} \frac{\left|\mu_i{}^{(a)}\right|^k}{k!} \leq p + \sum_{k\geq 1} \frac{1}{k!} \left(\sum_{i=1}^{p} \left|\mu_i{}^{(a)}\right|^k \right)$$

$$\leq p - 1 + \sum_{k\geq 0} \frac{\left(E_a(G)\right)^k}{k!}$$

$$EE_a(G) \leq p - 1 + e^{E_a(G)}.$$

Hence the result.

CONCLUSION:

In this paper, we have introduced some open neighbourhood degree-based reachability matrices of a graph and investigated their corresponding Energy and Estrada index. We further computed some open neighbourhood degree-based reachability energy for regular graphs. In addition, we have established bounds for both the Energy and the Estrada index derived from these matrices.

REFERENCES:

[1] Bala S, Vijay T, Thirusangu K, Reachability degree sum energy of graph, Journal of Tianjin university Science and Technology, Vol 57, Issue 05:2024, May, 325-334.

[2] Bala S, Vijay T, Thirusangu K, Energy variant and index in the context of connected graph, International Journal of Research Publication and Reviews, Vol 06, Issue 01, Jan 2025, 4746-4755.

[3] Bala S, Vijay T, Thirusangu K, Reachability degree sum: Energy and Estrada index of a graph, International Research Journal of Modernization in Engineering Technology and Science, Vol 07, Issue 02, Feb 2025, 566-577.

[4] Bala S, Vijay T, Thirusangu K, Energy and index in the family of reachability matrix for connected graph, IPE Journal of Management, vol 15, No. 12, January- June 2025, 175-196.

[5] Bondy J A, Murty U S R, Graph theory with applications, The Mac Millan Press LTD, London and Basingstoke,1979.

[6] De la Pe˜na J A, Gutman I, Rada J, Estimating the Estrada Index, Lin. Algebra Appl. 427 (2007) 70-76.

[7] Deng H, Radenkovic S, Ivan Gutman, The Estrada Index, In: Cvetkovic D, I Gutman (Eds.,), Applications of Graph Spectra, Math. Inst., Belgrade,2009, pp 123-140.

[8] Estrada E, Characterization of 3D Molecular Structure, Chem. Phys. Lett. 319 (2000) 713-718.

[9] Estrada E, Characterization of the Folding Degree of Proteins, Bioinformatics 18 (2002) 697-704.

[10] Estrada E, Characterization of Amino Acid Contribution to the Folding Degree of Proteins, Proteins 54 (2004) 727-737.

[11] Estrada E, Rodr´ıguez-Vel´azguez J A, Subgraph Centrality in Complex Networks, Phys. Rev. E 71 (2005) 056103-056103-9.

[12] Estrada E, Rodr´ıguez-Vel´azguez J A, Spectral Measures of Bipartivity in Complex Networks, Phys. Rev. E 72 (2005) 046105-146105-6.

[13] Estrada E, Rodr´ıguez-Vel´azguez J A, Randi´c M, Atomic Branching in Molecules, Int. J. Quantum Chem. 106 (2006) 823-832.

[14] Ivan Gutman, The energy of a graph, ber.math. Statist. Sekt. Forschangsz. Graz. 103,1-22,1978.

[15] Ivan Gutman, The Energy of a Graph: Old and New Results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196-211.

[16] Jahanbani A, Improve some of bounds for the Randic Estrada index of graphs, preprints, Dec 2020.

[17] Janezic D, Milicevic A, Nikolic S, Trinajstic, Graph theoretical matrices in chemistry, Univ. Kragujevac, Kragujevac, 2007.

[18] Ramane H S, Revankar D S, Patil J B, Bounds for the Degree sum eigenvalues and Degree sum Energy of a Graph, International journal of Pure and applied mathematical sciences, 6(2), 161-167, 2013.

[19] Sridhara G, Rajesh Kanna M.R, Parashivamurthy H.L, Energy of graphs and its new bounds, South East Asian Journal of Mathematics and Mathematical sciences, Vol. 18, N0.2 (2022), 161-170.