PAIR MEAN CORDIAL LABELING SOME FAMILIES OF SPECIAL GRAPHS

[1]A. Thirilogasundhari , [2]Dr. K. Balasangu
[1&2]Thiru kolanjiappar Government Arts College (Grade – I), Vriddhachalam- 606 001,
sundharinov93@gmail.com, balasangu76@gmail.com

ABSTRACT:

In this paper we study Pair Mean Cordial (PMC) labeling is type of graph labeling in which each edge receives a label determined by the mean (average) of the integer labels assigned to its endpoints and the resulting edge labels are distributed as evenly as possible. We investigate the existence of PMC-labeling for several well-known special graphs like gear graphs, bistar graphs, book graphs, barbell graphs, and their unions. For each class, we determine necessary and sufficient conditions under which a PMC-labeling exists, and provide constructions or counterexamples accordingly. We also examine how the PMC-property behaves under union operations and identify criteria that preserve cordiality.

1.INTRODUCTION:

Graph theory has established itself as a versatile and influential branch of mathematics, with far-reaching applications across science, engineering, and technological innovation. Within this domain, **graph labeling** has emerged as a particularly dynamic area of inquiry, valued for its ability to transform the structural features of graphs into numerical representations. This transformation involves assigning integers to the vertices, edges, or both, in accordance with specific rules or criteria. Labeling techniques serve as essential tools for modeling complex systems and addressing practical challenges in fields such as computer science, electrical engineering, and data communication.

In this work, we focus exclusively on finite, undirected, and simple graphs. Let G = (V, E) represent such a graph, where V is the vertex set and E is the edge set. The **order** of the graph defined as |V| = p representing the total number of vertices, while the **size** of the graph is |E|=q, indicating the total number of edges.

The study of graph labeling can be traced back to the ground-breaking contribution of Rosa in 1967 [1], who introduced the concept of **graceful labeling**. In this innovative scheme, vertices are labeled with distinct integers from the set {0, 1, ..., q}, such that the absolute differences between labels of adjacent vertices (used as edge labels) are all distinct and cover the range from {1 to q}. Rosa's formulation laid the cornerstone for a rich and rapidly growing research area, inspiring a multitude of labeling strategies over the years.

Subsequently, Harary [2] played a pivotal role in standardizing the terminology and symbols used in graph theory, providing a unified language for researchers. Gallian's and frequently updated survey [3] continues to document the evolution of graph labeling methods and their applications, underscoring the sustained interest in this field.

A notable development in this area was the introduction of **cordial labeling** by Cahit in 1987 [4], which provided a more inclusive alternative to the stricter frameworks of graceful and harmonious labeling. In cordial labeling, vertices are assigned binary values (0 or 1), and the labeling is considered cordial if the number of vertices receiving each label differs by at most one, and the same balance condition applies to edge labels derived from the sum modulo 2 of adjacent vertex labels. This relaxed criterion allowed a broader class of graphs to qualify as cordially labelled.In line with these advancements, we have recently worked a novel labeling technique termed **Pair Mean Cordial Labeling (PMC-labeling)** concept introduced by R. Ponraj [5]. This innovative approach combines elements of cordial labeling with arithmetic mean principles, offering new perspectives and analytical tools for studying specific graph families.

2. DEFINITION:

In this section we discussed about basic definitions for the special graphs.

Definition 1: The gear graph G_n arises by inserting intermediate vertices between every two successive nodes on a cycle, and then establishing connections from all peripheral and inserted nodes to a central core vertex. G_n has 2n+1 vertices and 3n edges.

Definition 2: A bistar graph $B_{m,n}$ is constructed by joining the centers of two star graphs S_m and S_n where center u with m leaves u_1 , u_2,…., u_m ; center v with n leaves v_1 , v_2,…., v_n and then the centers u and v are connected by an edge.

Definition 3: The book graph be V (B_M) = $\{ (v,i) | v \in V(S_{M+1}), i \in \{1,2\}\}$, thus B_M Has 2m+2 vertices and 3m+1 edges where m edges in each copy of S_{M+1} And m+1 vertical edges from the Cartesian product.

Definition 4: The barbell graph which consists of two complete graphs K_n. Let the vertex set be $V(B_n) = \{u_1, u_2, \dots, u_n \cup v_1, v_2, \dots, v_n\}$ thus labeling set as $(\frac{2n}{2})$ and edge set $E(B_n)$ contains all edges within each K_n : $\binom{n}{2}$ edges from u_n and $\binom{n}{2}$ from v_n edge labeling set as 2n +1.

3. PAIR MEAN CORDIAL GRAPH FOR SOME SPECIAL GRAPHS.

In this section, we parleyed about the concept of pair mean cordial graph and present associated theorems and their proofs.

Definition 5: Let G = (V,E) be a graph with p vertices and q edges. Define $\rho = \begin{cases} \frac{p}{2} & \text{p is even} \\ \frac{p-1}{2} & \text{p is odd} \end{cases}$, and $M = \{\pm1, \pm2, \dots, \pm\rho\}$. Consider a mapping $\lambda: V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to p − 1 elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge (uv) of G, there exists a labeling If $\lambda(u) + \lambda(v)$ is even the edge label = $\frac{\lambda(u)+\lambda(v)}{2}$ .If is odd, edge label = $\frac{\lambda(u)+\lambda(v)+1}{2}$ such that $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| \leq 1$. where $\overline{\mathbb{R}}_{\lambda}$ *and* $\overline{\mathbb{R}}_{\lambda^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling is called a pair mean cordial graph.

Theorem 1: The gear graph G_n is a pair mean cordial graph for all $n \geq 3$.

Proof: Let G_n be the gear graph obtained by inserting a new vertex between every pair of adjacent vertices in the outer of the wheel graph W_n. Let $C_n = v_1, v_2, \dots\dots, v_n$ be the cycle vertices. Let u_i be the inserted vertex between v_i and v_{i+1} with indices modulo n and O be the central vertex. The number of vertices $|V(G_n)| = 2n + 1$. number of edges $|E(G_n)| = 3n$. we defined a vertex labeling $\lambda : V(G_n) \to \mathbb{Z}$.

- If $\lambda(u) + \lambda(v)$ is even → the edge label = $\frac{\lambda(u)+\lambda(v)}{2}$.
- If odd: edge label = $\frac{\lambda(u)+\lambda(v)+1}{2}$.

Case 1: n is even

$2n + 1$ is odd by the PMC definition, n distinct labels from a set of consecutive integers and we repeat a vertex label. The vertex label from $\{-n, -(n-1), \ldots -1, 1, \ldots n\}$.

Case 2: n is odd

$2n+1$ is also odd by the PMC definition, n distinct labels from a set of consecutive integers and we repeat a vertex label. The vertex label from $\{-n, -(n-1), \ldots -1, 1, \ldots n\}$. In both even and odd cases a labeling exists and gives perfect edge labeling with distinct values from 1 to 3n. Therefore gear graph G_n is a PMC-graph for every $n \geq 3$.

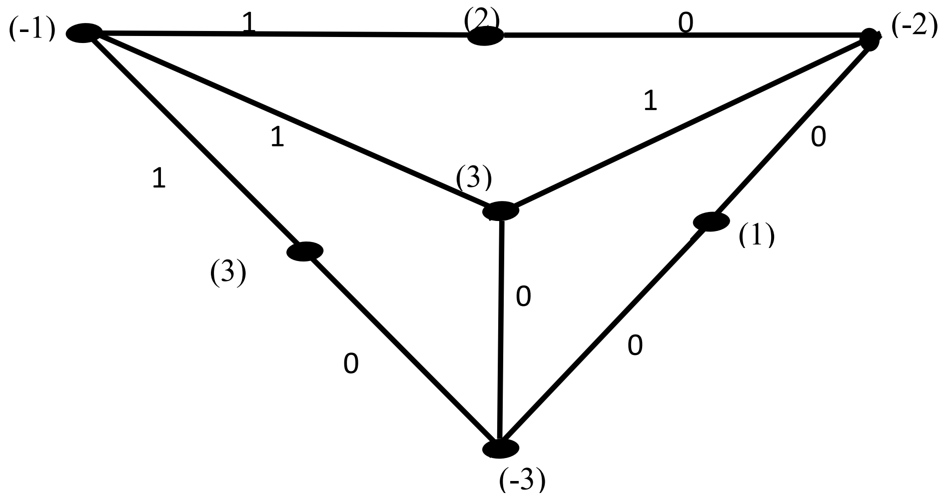

Figure:1

Theorem 2: Let $B_{m,n}$ be a bistar graph is a PMC-graph for all $(m,n) \geq 2$.

Proof: A bistar graph $B_{m,n}$ is constructed by joining the centers of two star graphs S_m and S_n where center u with m leaves u_1 , $u_2, \ldots, u_m$; center v with n leaves v_1 , $v_2, \ldots, v_n$ and then the centers u and v are connected by an edge. $B_{m,n}$ admits a labeling $L : V(B_{m,n}) \rightarrow \mathbb{Z}$ such that the induced edge labels. Let $V(B_{m,n}) = \{u\} \cup \{v\} \cup \{u_i \mid 1 \leq i \leq m\} \cup \{v_j \mid 1 \leq i \leq n\}$ and edge $E(B_{m,n}) = \{uu_i \mid 1 \leq i \leq m\} \cup \{vv_j \mid 1 \leq i \leq n\} \cup \{uv\}$.

Case 1: Both (m,n) are even

Assign distinct integer labels to the leaves of each star such that the sum of labels on edges adjacent to u and v take distinct values. Let the labels to leaves of S_m and the labels to leaves of S_n from $\frac{-(m+n)}{2}, \ldots -1, 1, \ldots, \frac{(m+n)}{2}$.

Case 2: Both (m,n) are odd

Suppose m is odd, Assign distinct integer labels to the leaves of each star such that the sum of labels on edges adjacent to u and v take distinct values. Let the labels to leaves of S_m and the labels to leaves of S_n from $\frac{-(m+n)}{2}$,…-1,1,…,$\frac{(m+n)}{2}$. consequently $\overline{\mathbb{R}}_\lambda = n - 1$ and $\overline{\mathbb{R}}_{\lambda^c} = n$. where $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_\lambda| \leq 1$. hence $B_{m,n}$ is a PMC-graph for all (m,n) ≥ 2.

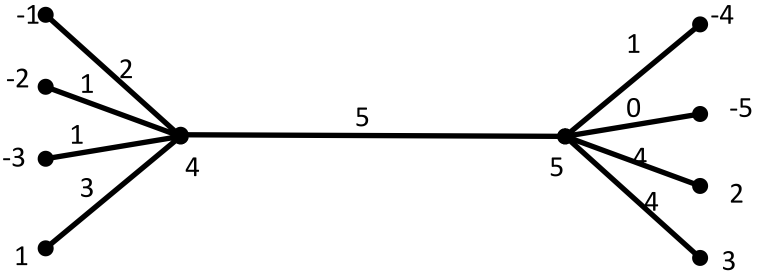

Figure : 2

Theorem 3: The book graph B_m is pair mean cordial graph for all m ≥ 3.

Proof: Let the vertex set of the book graph be V (B_m) = { (v, i)| v ∈ V(S_{m+1}), i ∈ {1,2}}, thus B_m has 2m+2 vertices and 3m+1 edges where m edges in each copy of S_{m+1} and m+1 vertical edges from the Cartesian product. Let m = 2k for k ≥ 1, defined a vertex labeling λ as $\lambda(u) + \lambda(v)$ is even edge label = $\frac{\lambda(u)+\lambda(v)}{2}$ and odd = $\frac{\lambda(u)+\lambda(v)+1}{2}$; then $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_\lambda| \leq 1$. Let we assign the odd vertices as {-1 , -2 , -3 and -4} for v_1, v_3, v_5 and v_7 and for even vertices v_2, v_4, v_6 and v_8 as {3 , 4 , 2, 1}. Then $\overline{\mathbb{R}}_\lambda = n$ and $\overline{\mathbb{R}}_{\lambda^c} = n$. where $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_\lambda| = 0$ which is ≤ 1. Hence The book graph is pair mean cordial is satisfied for all m ≥ 3.

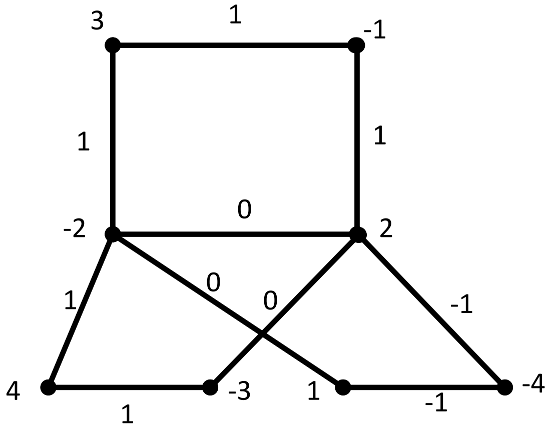

Figure : 3

Theorem 4: The barbell graph B_n is pair mean cordial (PMC) graph for all n ≥ 3.

Proof: Let B_n denoted as barbell graph which consists of two complete graphs K_n. Let the vertex set be $V(B_n) = \{u_1, u_2, \ldots, u_n \cup v_1, v_2, \ldots, v_n\}$ thus labeling set as $(\frac{2n}{2})$ and edge set $E(B_n)$ contains all edges within each K_n : $\binom{n}{2}$ edges from u_n and $\binom{n}{2}$ from v_n edge labeling set as 2n +1. Let defined $\lambda : V \to M = \{\pm 1, \pm 2, \ldots \pm \frac{2n}{2}\}$ by assign different label in M to element of V.

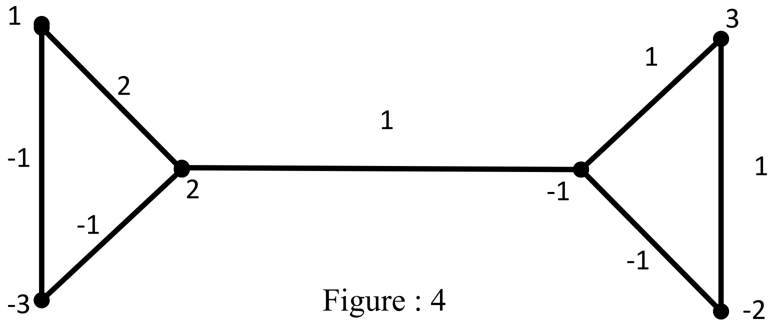

Figure : 4

Now the label of vertices of $\{u_1, u_2, \ldots, u_n \cup v_1, v_2, \ldots, v_n\}$ assign the odd vertices as {1 , 2, 3} for v_1, v_3, v_5 and for even vertices v_2, v_4, v_6 as {-3 , -1 , -2}. Then $\overline{\mathbb{R}}_\lambda = n - 1$ and $\overline{\mathbb{R}}_{\lambda^c} = n$. where $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_\lambda| = 4 - 3$ which is ≤ 1. Hence The barbell graph is pair mean cordial is satisfied for all n ≥ 3.

4. UNION 0F THE GRAPHS:

The amalgamation of two graphs G_1 and G_2 results in a new graph, represented as($G_1 \cup G_2$) which incorporates all the vertices and connections present in both original graphs.

Theorem 5: If G_n be a gear graph and $B_m$be book graph, the union of the two graphs ($G_n \cup B_m$) is a pair mean cordial graph where n = m, m≥ 3 .

Proof: The gear graph G_n is constructed by adding a new vertex between each pair of consecutive vertices on the outer cycle of the wheel graph W_n. So G_n has 2n+1 vertices and 3n edges. the vertex set of the book graph be V (B_m) = { $(v, i) | v \in V(S_{m+1}), i \in \{1,2\}$}, thus B_m has 2m+2 vertices and 3m+1 edges where m edges in each copy of S_{m+1} and m+1 vertical edges from the Cartesian product. Now the union of two graphs ($G_n \cup B_m$) has transformed to 4n + 1 vertices and 6n edges. Let $\lambda : V(G_n \cup B_m) \to M = \{\pm 1, \pm 2, \ldots \pm n\}$ Assign labels to the vertices from a set of n distinct integers, allowing one label to be used twice. Based on this vertex labeling, define an induced labeling for each edge according to the values assigned to its endpoints ∈ E , defined as

$$\lambda^*(uv) = \begin{cases} \frac{\lambda(u)+\lambda(v)}{2}, & \text{if } \lambda(u)+\lambda(v) \text{is even} \\ \frac{\lambda(u)+\lambda(v)+1}{2}, & \text{if } \lambda(u)+\lambda(v) \text{is odd} \end{cases}, \quad \text{then } |\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| \leq 1.$$

Let we assign the odd vertices as {-1 , -2 , -3 , -4 , -5, -6, 6 } for (v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}) and for even vertices ($v_2, v_4, v_6, v_8, v_{13}, v_{14}$ and v_{15}) as {3,4 ,5, (5) ,1,2 }.

$\overline{\mathbb{R}}_{\lambda} \rightarrow$ (be the number of edges labeled as 1) = n

$\overline{\mathbb{R}}_{\lambda^c} \rightarrow$ (be the number of edges not labeled as 1) = n .

where $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| = 9 - 9 \leq 1$. Therefore, the union of the gear graph and the book graph, ($G_n \cup B_m$) is well-defined and satisfies the required structural conditions.

Example:

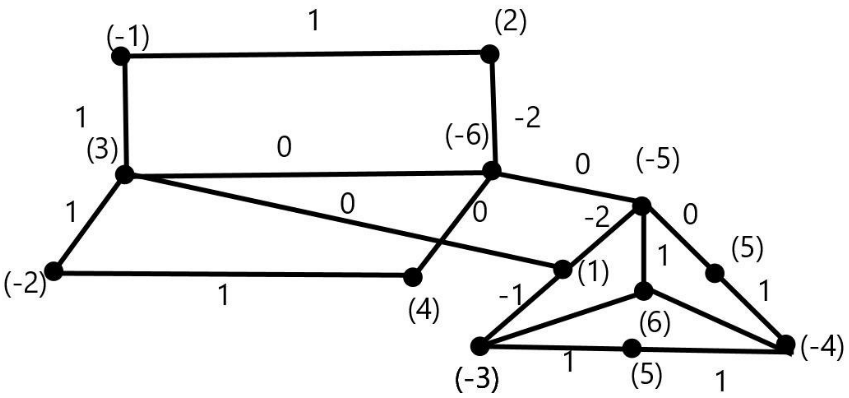

Figure:5

Theorem 6:

The union of the two graphs gear graph and barbell ($G_n \cup B_n$) a pair mean cordial graph where n = n, n≥ 3.

Proof: Let we note that the gear graph G_n arises by inserting intermediate vertices between every two successive nodes on a cycle, and then establishing connections from all peripheral and inserted nodes to a central core vertex.. So G_n has 2n+1 vertices and 3n edges. Let B_n denoted as barbell graph which consists of two complete graphs K_n. Let the vertex set be $V(B_n) = \{u_1, u_2, \ldots, u_n \cup v_1, v_2, \ldots, v_n\}$ thus labeling set as $(\frac{2n}{2})$ and edge set $E(B_n)$ contains all edges within each K_n : $\binom{n}{2}$ edges from u_n and $\binom{n}{2}$ from v_n edge labeling set as

2n +1. Now the union of two graphs $(G_3 \cup B_3)$ has transformed to (4n – 1) vertices. Let $\lambda :$ $V(G_n \cup B_n) \rightarrow M = \{\pm1, \pm2, \ldots \pm n\}$ Using a set of n unique integers, assign labels to the vertices such that one label is duplicated for a single vertex. Then, define the value of each edge based on the labels of its endpoints, resulting in an induced edge labeling $\in E$, defined as

$$\lambda^* (uv) = \begin{cases} \frac{\lambda(u)+\lambda(v)}{2} \ , \text{ if } \lambda(u) + \lambda(v) \text{ is even} \\ \frac{\lambda(u)+\lambda(v)+1}{2} \ , \text{if } \lambda(u) + \lambda(v) \text{ is odd} \end{cases} \quad \text{then } |\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| \leq 1.$$

Example:

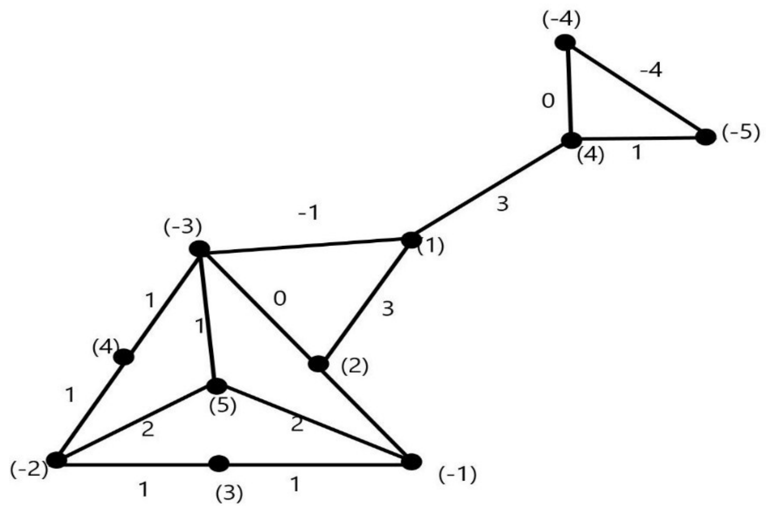

Figure: 6

Let we assign the vertices as {-1 , -2 , -3 , -4 , -5} for $(v_1, v_3, v_5, v_8, v_9)$ and for other vertices $(v_2, v_4, v_6, v_7, v_{10}$ and v_{11} as {3,4,1 , 4,2 and 5}.

$\overline{\mathbb{R}}_{\lambda} \rightarrow (\text{be the number of edges labeled as } 1) = n - 1$

$\overline{\mathbb{R}}_{\lambda^c} \rightarrow (\text{be the number of edges not labeled as } 1) = n$. where $|\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| = 7 - 8 \leq 1$. Hence The union of the gear graph and the barbell graph, ($G_3 \cup B_3$) adheres to the conditions outlined by the PMC framework therefore our hypothesis of ($G_n \cup B_n$) where n = n, n≥ 3 is a PMC..

Theorem 7: The union of the two graphs book and barbell graph ($B_m \cup B_n$) admits a pair mean cordial graph.

Proof: Let the vertex set of the book graph be V (B_m) = $\{ (v, i) | v \in V(S_{m+1}), i \in \{1,2\}\}$, thus B_m has 2m+2 vertices and 3m+1 edges where m edges in each copy of S_{m+1} and m+1 vertical edges from the Cartesian product. Let B_n denoted as barbell graph which consists of two complete graphs K_n. Let the vertex set be $V(B_n) = \{u_1, u_2, \ldots, u_n \cup v_1, v_2, \ldots, v_n\}$ thus labeling set as $(\frac{2n}{2})$ and edge set $E(B_n)$ contains all edges within each K_n : $\binom{n}{2}$ edges from u_n and $\binom{n}{2}$ from v_n edge labeling set as 2n +1. Now the union of two graphs $(B_m \cup B_n)$ has transformed to 4n vertices. Let $\lambda : V(B_m \cup B_n) \to M = \{\pm 1, \pm 2, \ldots \pm n\}$ as a set of n distinct integers and assign label has no repeat label. Now consider the induced edge labeling for each $\in E$, defined as

$$\lambda^* (uv) = \begin{cases} \frac{\lambda(u)+\lambda(v)}{2}, & \text{if } \lambda(u) + \lambda(v) \text{is even} \\ \frac{\lambda(u)+\lambda(v)+1}{2}, & \text{if } \lambda(u) + \lambda(v) \text{is odd} \end{cases}, \quad \text{then } |\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| \leq 1.$$

$$\overline{\mathbb{R}}_{\lambda} = n , \overline{\mathbb{R}}_{\lambda^c} = n$$

$$\text{where } |\overline{\mathbb{R}}_{\lambda^c} - \overline{\mathbb{R}}_{\lambda}| = 8 - 8 \leq 1.$$

Example:

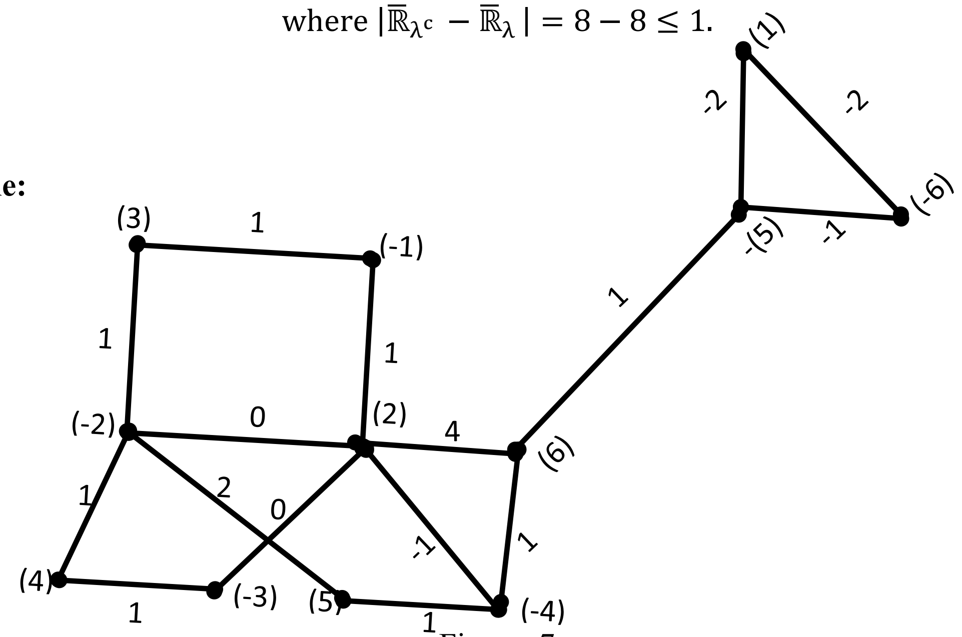

Figure :7

Hence, The union of the two graphs book graph and barbell graph ($B_3 \cup B_3$) is satisfied, therefore our hypothesis of ($B_m \cup B_n$) where n = m, m≥ 3 is a PMC.

RESEARCH OBJECTIVES:

This research aims to fill the existing gaps in the literature by:

1. Characterizing the conditions under which gear, bistar, book, and barbell graphs admit PMCL.
2. Developing constructive algorithms for assigning PMCL in these graphs.
3. Analyzing the behavior of PMCL in unions of these graph classes.

CONCLUSION:

In this paper the study of PMCL is specific expanding our understanding of graph labeling. By focusing on gear, bistar, book, and barbell graphs, as well as their unions, this research seeks to uncover new insights into the structural properties that facilitate PMCL and to develop algorithms for their labeling. The findings are expected to contribute significantly to the theoretical framework of graph labeling.

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