

# Graph of Pythagorean neutrosophic fuzzy ideal at near rings of $M$ gamma groups and its maximal product

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## Abstract:

Fuzzy sets have shown themselves to be a dependable subject of study in numerous domains. They are an excellent option for the simplification and extension of classical sets due to their simplicity and flexibility. This paper describes the maximal product of two Pythagorean Neutrosophic Fuzzy Ideal Graphs(PNFIG) and maximal product of two Strong Pythagorean Neutrosophic Fuzzy Ideal Graphs(SPNFIG) in the  $M\Gamma(\text{Gamma})$  group of near rings. The degree of a vertex is discussed in detail along with a few theorems for the maximal product of PNFI graph structures. The Pythagorean Neutrosophic Fuzzy Set [PNFS] is a generalization of the Pythagorean Fuzzy Set(PFS) with condition  $0 \leq \alpha_A(s)^2 + \beta_A(s)^2 + \gamma_A(s)^2 \leq 2$  and neutrosophic set with dependent neutrosophic components. Thus we extend some of the basic properties for this PNFIG along with few examples.

**Keywords:** Pythagorean Neutrosophic Fuzzy Ideal (PNFI), Regular graph of Pythagorean Neutrosophic Fuzzy Ideal, Graph of Pythagorean Neutrosophic Fuzzy Ideal.

## 1. Introduction

Fuzzy set theory was first proposed by L.A. Zadeh in 1965 [20]. As an extension of fuzzy sets, Atanassov [1] introduced a new set intuitionistic set that specifies elements with membership and non- membership grades, with the condition that the sum of both cannot be greater than one. Yager [19] provided a new view of pythagorean fuzzy sets, which have the constraint that their square sum is less than or equal to one. Pythagorean fuzzy graphs were created using the notion of pythagorean sets and are used in a variety of domains [7]. In 2005, Smarandache [17] created the neutrosophic set, which is a hybrid of intuitionistic and fuzzy sets.

A neutrosophic set has truth, falsity and indeterminacy membership for each element. In particular, the neutrosophic set has been applied to fuzzy graph theory and many new concepts

of neutrosophic fuzzy graph has been developed. In neutrosophic set, out of three dependency cases we choose one special case in which indeterminacy is independent, truth and falsity are completely dependent with  $0 \leq \alpha_A(s)^2 + \beta_A(s)^2 + \gamma_A(s)^2 \leq 2$  is termed as Pythagorean neutrosophic set. In this research, we put on the Pythagorean neutrosophic sets to neutrosophic fuzzy graphs.

A near-ring is an algebraic structure that resembles a ring and satisfies a small number of axioms. It is possible for groups with organic functions to create near-rings. Pilz et al. [14] have created a theory of near rings with many traits and characteristics based on differential equations, geometry, topology, and automation. Jianming et al. [3] integrated the concepts of intuitionistic fuzzy sets and near rings in 2005.

Non-membership values are indeterministic, which causes uncertainty, for the same reasons that membership values cannot always be estimated. Smarandache et al. [11, 17] proposed the Neutrosophic Set (NS), which is a generalization of conventional sets, fuzzy sets, intuitionistic fuzzy sets, and so on. Durgadevi et al. [2] describe the features of neutrosophic fuzzy (NF) objectives in  $\Gamma$  Rings. Some novel neutrosophic activities are examined.

Their applications range from automation, topology, and ideal theory to graph theory and order theory. Rosenfeld et al. [15] expanded graphs to fuzzy graphs in 1975 and described their use in decision making. Satyanarayana and Prasad et al. [16] used theorems and graphs to enlarge near rings with fuzzy ideals and theoretical knowledge of graph theory. Kim and Kim et al. [5] proposed fuzzy ideals for NR in 1996, and Jun et al. [4] extended them to FNR. Zhan et al. [21] accepted the theory of IF ideals of nearby rings and deduced certain associated features. Palaniappan et al. [13] defined fuzzy ideals of gamma rings in IF sets as extensions of gamma ring ideals. IF graphs were divided into strongest arcs, strongest paths, weakest paths, alpha- strong, beta-strong, and delta-weak arcs in a study by Karunambikai et al. [6].

In the near rings, Mala and Shanmugapriya have examined several features of IF ideals of  $M$  gamma groups [8]. A generalization of the concept of fuzzy ideal graphs of semi groups was made by Murali Krishna Rao et al. [12].  $M\Gamma$  groups were given a graph of intuitionistic fuzzy ideals by Mala and Shanmugapriya [9]. Also Mala and Shanmugapriya [10] were introduced the concept of maximal product of two graph of intuitionistic fuzzy ideals of  $M\Gamma$  groups in near ring. Udayanatchi and Shanmuga Sundaram [18] introduced the concept of graph of pythagorean neutrosophic fuzzy ideal of  $M\Gamma$  Gamma groups in near rings. This paper describes the maximal product of two pythagorean neutrosophic fuzzy (NF) graphs and two pythagorean neutrosophic fuzzy (NF) graph in the  $M\Gamma$  group of near rings. The degree of a vertex is discussed in detail along with a few theorems for the maximal product of PNF graph structures.

## 2. Preliminaries

A few key definitions and necessary outcomes are provided in this section.

### 2.1 Ideal

Let a ring be  $(R, +, \cdot)$ . A non-empty subset  $I$  of  $R$  is termed Ideal if,

- (i) For all  $a, b \in I$  then  $-b \in I$ .
- (ii) For all  $a \in I, r \in R$  then  $ar \in I$  (Right Ideal)
- (iii) For all  $a \in I, r \in R$  then  $ra \in I$  (Left Ideal)

### 2.2 Fuzzy $M\Gamma$ - Subgroup of $M\Gamma$ group

If a fuzzy set  $T$  of  $G$  meets both of the following criteria, it is referred to as a fuzzy  $M\Gamma$  subgroup of  $G$ :

- (i)  $\alpha(a - b) \geq \max\{\alpha(a), \alpha(b)\}$  and
- (ii)  $\alpha(a \nu b) \geq \alpha(b)$  for all  $b \in G, a \in M$  and  $\nu \in \Gamma$ . Here,  $M$  symbolizes a  $\Gamma$  near ring and  $G$  stands for an  $M\Gamma$  group.

### 2.3 Neutrosophic Fuzzy Ideal of $M\Gamma$ Groups in Near Rings ( $NFI\ M\Gamma\ GNR$ )

A NF set  $\mathcal{N}(\alpha_I, \beta_I, \gamma_I)$  of  $G^*$  in  $N^*$  is stated as  $NFIM\Gamma SG$  in  $G^*$  if it agrees the following two sets inequalities.

- $\alpha_I(a - b) \geq \max\{\alpha_I(a), \alpha_I(b)\}$   
 $\beta_I(a - b) \leq \min\{\beta_I(a), \beta_I(b)\}$   
 $\gamma_I(a - b) \leq \min\{\gamma_I(a), \gamma_I(b)\}$
- $\alpha_I(t \nu b) \geq \alpha_I(b)$   
 $\beta_I(t \nu b) \leq \beta_I(b)$   
 $\gamma_I(t \nu b) \leq \gamma_I(b)$  for all  $a, b \in G^*, t \in N^*$  and  $\nu \in \Gamma$ .

Where  $N^*$  indicates  $\Gamma NR$  and  $G^*$  for a  $M\Gamma G$ .

### 2.4 Pythagorean Neutrosophic Fuzzy Ideal of $M\Gamma$ Groups in Near Rings( $PNFIM\Gamma GNR$ )

Let  $N$  be a near ring. Let  $G^*$  be an  $M\Gamma$  group over a near ring  $N$ . A  $PNFI\ I_p$  of the  $M\Gamma$  - group  $G^*$  is a mapping  $I_p: G^* \rightarrow [0,1] \times [0,1] \times [0,1]$  such that for each element  $a \in G^*, I_p(a) = (\alpha_I(a), \beta_I(a), \gamma_I(a))$ , where  $\alpha_I(a)$  signifies the degree of membership,  $\beta_I(a)$

signifies the degree of indeterminacy and  $\gamma_I(a)$  represents the degree of non-membership and the following condition holds for all  $a \in G^*$ :  $(\alpha_I(a))^2 + (\beta_I(a))^2 + (\gamma_I(a))^2 \leq 2$ .

Additionally, the mapping  $I$  satisfies

- $\alpha_I(a - b) \geq \max\{\alpha_I(a), \alpha_I(b)\}$   
 $\beta_I(a - b) \leq \min\{\beta_I(a), \beta_I(b)\}$   
 $\gamma_I(a - b) \leq \min\{\gamma_I(a), \gamma_I(b)\}$
- $\alpha_I(t \nu b) \geq \alpha_I(b)$   
 $\beta_I(t \nu b) \leq \beta_I(b)$   
 $\gamma_I(t \nu b) \leq \gamma_I(b)$  for all  $a, b \in G^*, t \in N^*$  and  $\nu \in \Gamma$ .

Where  $N^*$  indicates  $\Gamma NR$  and  $G^*$  for a  $MTG$ . Then  $I_P$  is called PNFI of the  $MT$ -group in the near-ring  $N$ .

### 2.5 Graph of a Near Ring with Respect to an Ideal

Let  $I$  be  $N$ 's ideal. In the graph of  $N$  with respect to  $I$ , every member of  $N$  is a vertex, and an edge connects two distinct vertices,  $a$  and  $b$ , if and only if  $aNb \subseteq I$  or  $bNa \subseteq I$ . The graph of  $N$  with respect to  $I$  is denoted by  $G_I(N)$ .

### 2.6 Graph of the Pythagorean Fuzzy ideal(PFI)

A Pythagorean fuzzy ideal graph(PFIG) is  $G_P = (V_I, E_I)$  with  $\alpha_I$  and  $\beta_I$  from  $V_I$  to  $[0,1]$  signifying membership and non-membership functions of  $V_I$  such that

$0 \leq \alpha_I^2(a) + \beta_I^2(a) \leq 1$  for all  $a \in V_I$  and  $E_I \subseteq V_I \times V_I$  with  $\alpha_I$  and  $\beta_I$  from  $V_I \times V_I$  to  $[0,1]$  such that

- $\alpha_I(ab) \geq \max\{\alpha_I(a), \alpha_I(b)\}$
- $\beta_I(ab) \leq \min\{\beta_I(a), \beta_I(b)\}$  with  $0 \leq \alpha_I^2(ab) + \beta_I^2(ab) \leq 1$  for all  $ab \in E_I$

The graph of the PFI ideal  $I_P(\alpha_I, \beta_I)$  is thus called  $G_P(V_I, E_I)$  and is described as  $G_P(V_I, E_I, \alpha_I, \beta_I)$

## 3. Graph of a Pythagorean Neutrosophic Fuzzy Ideals of $MT$ Group in Near Rings and its Maximal Product

A product of two graphs can be obtained as an extension of graph using fuzzy and Pythagorean neutrosophic fuzzy sets. The new idea of maximal product structure of PNFIMTG is defined and their properties are discussed.

### 3.1 Graph of the Pythagorean Neutrosophic Fuzzy ideal(PNFI)

A Pythagorean fuzzy ideal graph(PFIG) is  $G_P(V_I, E_I)$  with  $\alpha_I$  and  $\beta_I$  from  $V_I$  to  $[0,1]$  signifying membership and non-membership functions of  $V_I$  such that  $0 \leq \alpha_I^2(a) + \beta_I^2(a) \leq$

1 for all  $a \in V_I$  and  $E_I \subseteq V_I \times V_I$  with  $\alpha_I$  and  $\beta_I$  from  $V_I \times V_I$  to  $[0,1]$  such that

- $\alpha_I(ab) \geq \max\{\alpha_I(a), \alpha_I(b)\}$
- $\beta_I(ab) \leq \min\{\beta_I(a), \beta_I(b)\}$
- $\gamma_I(ab) \leq \min\{\gamma_I(a), \gamma_I(b)\}$  with  $0 \leq \alpha_I^2(ab) + \beta_I^2(ab) + \gamma_I^2(ab) \leq 2$ , for all  $ab \in E_I$

Then  $G_P(V_I, E_I)$  is characterized as  $G_P(V_I, E_I, \alpha_I, \beta_I, \gamma_I)$  and is stated to as the graph of the PNF ideal  $I_P(\alpha_I, \beta_I, \gamma_I)$ .

### 3.2 Definition

Let  $G_{P_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{P_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  be two graphs of PNFIM $\Gamma$ GNR  $I_1$  and  $I_2$  in near ring  $N^*$  then  $G_{P_1} * G_{P_2} = (V_I, E_I, \alpha_I, \beta_I, \gamma_I)$  is called maximal product structure of PNFIM $\Gamma$ GNR. The set of vertices  $V_I = V_{I_1} \times V_{I_2}$  exist with

$$\alpha_I(a_1, b_1) = \alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1),$$

$$\beta_I(a_1, b_1) = \beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1) \text{ and}$$

$$\gamma_I(a_1, b_1) = \gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1) \text{ such that}$$

$$0 \leq [\alpha_I(a_1, b_1)]^2 + [\beta_I(a_1, b_1)]^2 + [\gamma_I(a_1, b_1)]^2 \leq 2 \text{ for all } (a_1, b_1) \in V_I.$$

The set of edges

$$E_I = \{((a_1, b_1)(a_2, b_2)) / a_1 = a_2 \text{ and } b_1, b_2 \in E_{I_2} \text{ (or) } b_1 = b_2 \text{ and } a_1, a_2 \in E_{I_1}\}$$

exist with

$$\alpha_I((a_1, b_1)(a_2, b_2)) = \{\alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \text{ \& } b_1 b_2 \in E_{I_2}$$

$$\alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \text{ \& } a_1 a_2 \in E_{I_1}$$

$$\beta_I((a_1, b_1)(a_2, b_2)) = \{\beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \text{ \& } b_1 b_2 \in E_{I_2}$$

$$\beta_{I_2}(b_2) \vee \beta_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \text{ \& } a_1 a_2 \in E_{I_1} \text{ and}$$

$$\gamma_I((a_1, b_1)(a_2, b_2)) = \{\gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \text{ \& } b_1 b_2 \in E_{I_2}$$

$$\gamma_{I_2}(b_2) \vee \gamma_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \text{ \& } a_1 a_2 \in E_{I_1}$$

such that  $0 \leq [\alpha_I((a_1, b_1)(a_2, b_2))]^2 + [\beta_I((a_1, b_1)(a_2, b_2))]^2 + [\gamma_I((a_1, b_1)(a_2, b_2))]^2 \leq 2$

Here  $E_I$  (edges set) has edges only if either the first coordinates are same, or the second coordinates are same with an edge existing already in  $G_{P_1}$  or  $G_{P_2}$ .

### 3.3 Example

Consider Pythagorean neutrosophic fuzzy graph of the ideal  $I_1 = \{0\}$  and  $I_2 = \{0,1\}$  of  $Z_2$  as  $G_{P_1}$  and  $G_{P_2}$  then, we get  $G_{P_1} * G_{P_2}$  as a maximal product of graphs of PNFIM $\Gamma$ G as follows:

$G_{P_1} * G_{P_2}$  has vertex set  $V_I = V_{I_1} \times V_{I_2} = \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$  and edge set  $E_I$

has edges with either first co-ordinate same or second co-ordinate same in  $V_I$  along with the remaining coordinates exists as edge in their corresponding graph. Thus  $G_{P_1} * G_{P_2}$  is represented as follows:



Figure 1: Graph of  $G_{P_1}$



Figure 2: Graph of  $G_{P_2}$

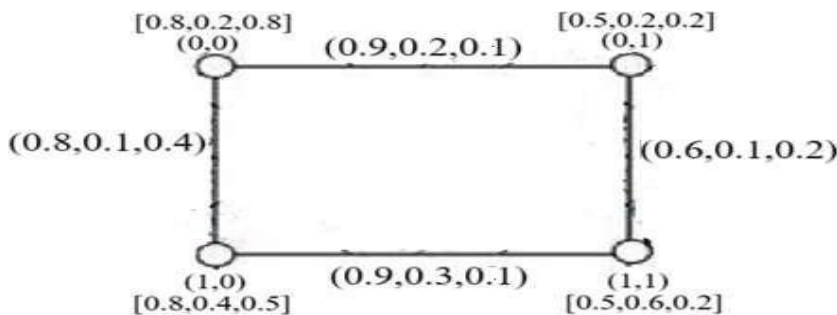


Figure 3 : Graph of  $G_{P_1} * G_{P_2}$

### 3.4 Definition

The vertex degree of maximal product graph structures  $G_{P_1} * G_{P_2}$  of PNFIM $\Gamma$ GNR,

$G_{P_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{P_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  is given by:

$$D_{G_{P_1} * G_{P_2}} \alpha_I(a_j, b_j) = \sum \alpha_{I_1}(a_i a_k) \vee \alpha_{I_2}(b_j) + \sum \alpha_{I_2}(b_j b_i) \vee \alpha_{I_1}(a_i)$$

$$D_{G_{P_1} * G_{P_2}} \beta_I(a_j, b_j) = \sum \beta_{I_1}(a_i a_k) \wedge \beta_{I_2}(b_j) + \sum \beta_{I_2}(b_j b_i) \wedge \beta_{I_1}(a_i) \text{ and}$$

$$D_{G_{P_1} * G_{P_2}} \gamma_I(a_j, b_j) = \sum \gamma_{I_1}(a_i a_k) \wedge \gamma_{I_2}(b_j) + \sum \gamma_{I_2}(b_j b_i) \wedge \gamma_{I_1}(a_i)$$

Therefore, from figure 3, degree of all vertices of  $G_{P_1} * G_{P_2}$  can be calculated as follows:

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(1,0) &= \alpha_{I_1}(1 \ 0) \vee \alpha_{I_2}(0) + \alpha_{I_2}(0 \ 1) \vee \alpha_{I_1}(1) \\ &= (0.6 \vee 0.8) + (0.9 \vee 0.3) \end{aligned}$$

$$= 0.8 + 0.9 = 1.7$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(1,0) &= \beta_{I_1}(1\ 0) \wedge \beta_{I_2}(0) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(1) \\ &= (0.1 \wedge 0.4) + (0.3 \wedge 0.7) \\ &= 0.1 + 0.3 = 0.4 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(1,0) &= \gamma_{I_1}(1\ 0) \wedge \gamma_{I_2}(0) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(1) \\ &= (0.4 \wedge 0.9) + (0.1 \wedge 0.5) \\ &= 0.4 + 0.1 = 0.5 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(0,0) &= \alpha_{I_1}(0\ 1) \vee \alpha_{I_2}(0) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(0) \\ &= (0.6 \vee 0.8) + (0.9 \vee 0.4) \\ &= 0.8 + 0.9 = 1.7 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(0,0) &= \beta_{I_1}(0\ 1) \wedge \beta_{I_2}(0) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(0) \\ &= (0.1 \wedge 0.4) + (0.3 \wedge 0.2) \\ &= 0.1 + 0.2 = 0.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(0,0) &= \gamma_{I_1}(0\ 1) \wedge \gamma_{I_2}(0) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(0) \\ &= (0.4 \wedge 0.9) + (0.1 \wedge 0.8) \\ &= 0.4 + 0.1 = 0.5 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(1,1) &= \alpha_{I_1}(1\ 0) \vee \alpha_{I_2}(1) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(1) \\ &= (0.6 \vee 0.5) + (0.2 \vee 0.3) \\ &= 0.5 + 0.3 = 0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(1,1) &= \beta_{I_1}(1\ 0) \wedge \beta_{I_2}(1) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(1) \\ &= (0.1 \wedge 0.6) + (0.7 \wedge 0.7) \\ &= 0.6 + 0.7 = 1.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(1,1) &= \gamma_{I_1}(1\ 0) \wedge \gamma_{I_2}(1) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(1) \\ &= (0.4 \wedge 0.2) + (0.1 \wedge 0.5) \\ &= 0.2 + 0.1 = 0.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(0,1) &= \alpha_{I_1}(1\ 0) \vee \alpha_{I_2}(1) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(0) \\ &= (0.6 \vee 0.5) + (0.9 \vee 0.4) \\ &= 0.6 + 0.9 = 1.5 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(0,1) &= \beta_{I_1}(1\ 0) \wedge \beta_{I_2}(1) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(0) \\ &= (0.1 \wedge 0.6) + (0.3 \wedge 0.2) \end{aligned}$$

$$= 0.1 + 0.2 = 0.3$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(0,1) &= \gamma_{I_1}(1,0) \wedge \gamma_{I_2}(1) + \gamma_{I_2}(0,1) \wedge \gamma_{I_1}(0) \\ &= (0.4 \wedge 0.2) + (0.1 \wedge 0.8) \\ &= 0.2 + 0.1 = 0.3 \end{aligned}$$

### 3.5 Example

Consider Pythagorean neutrosophic fuzzy graph of the ideal  $I_1 = \{0\}$  and  $I_2 = \{0,1,2\}$  of  $Z_3$  as  $G_{P_1}$  and  $G_{P_2}$  then, we get  $G_{P_1} * G_{P_2}$  as a maximal product of graphs of PNFIMFG as to follows:  $G_{P_1} * G_{P_2}$  has vertex set

$$V_I = V_{I_1} \times V_{I_2} = \{0,1,2\} \times \{0,1,2\} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

and Each edge set  $E_I$  has edges with a similar first coordinate or a similar second coordinate as the edge set  $V_I$  along with the remaining coordinates. Thus  $G_{P_1} * G_{P_2}$  is characterized as follows:

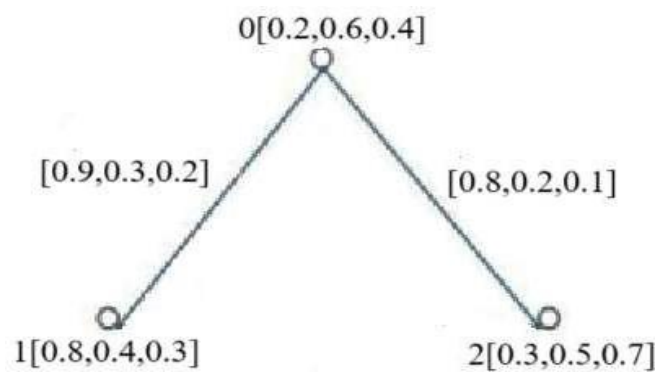


Figure 4 : Graph of  $G_{P_1}$

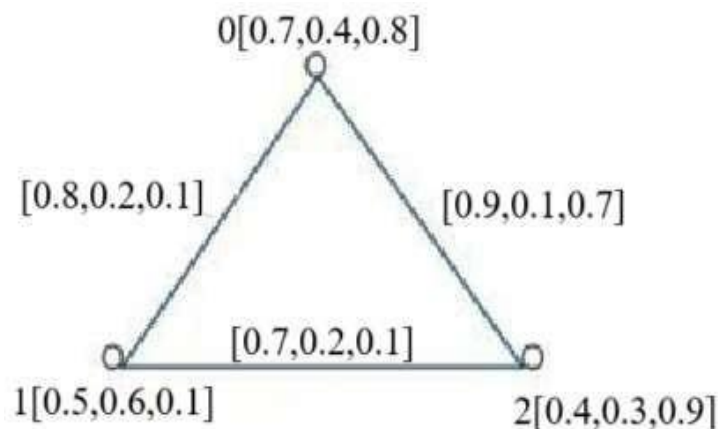


Figure 5 : Graph of  $G_{P_2}$

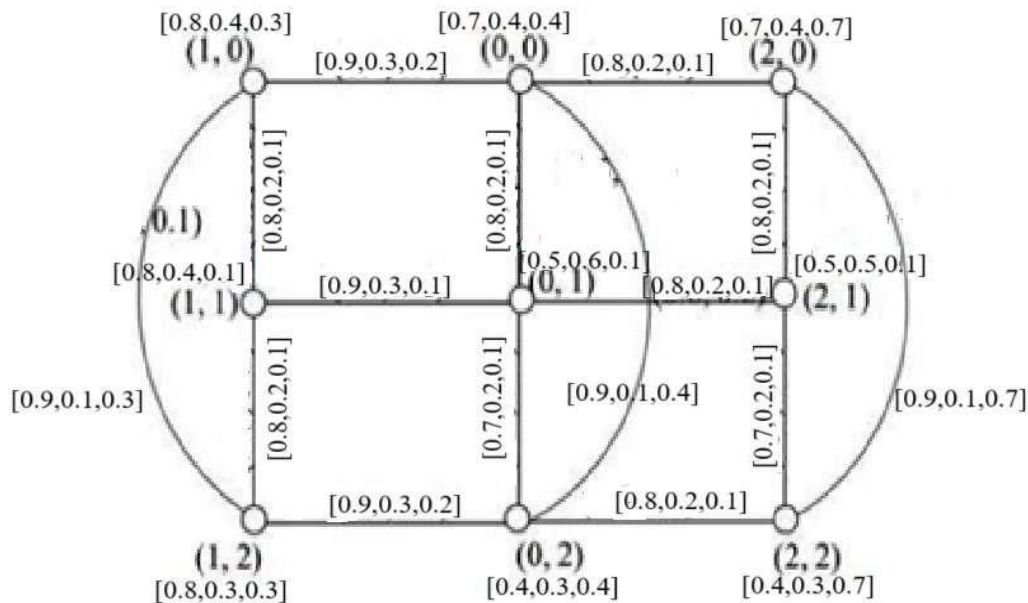


Figure 6 : Graph of  $G_{P_1} * G_{P_2}$

**3.6 Definition**

The vertex degree of maximal product graph structures  $G_{P_1} * G_{P_2}$  of PNFIM $\Gamma$ GNR,

$G_{P_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{P_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  is given by:

$$D_{G_{P_1} * G_{P_2}} \alpha_I(a_j, b_j) = \sum \alpha_{I_1}(a_i a_k) \vee \alpha_{I_2}(b_j) + \sum \alpha_{I_2}(b_j b_i) \vee \alpha_{I_1}(a_i)$$

$$D_{G_{P_1} * G_{P_2}} \beta_I(a_j, b_j) = \sum \beta_{I_1}(a_i a_k) \wedge \beta_{I_2}(b_j) + \sum \beta_{I_2}(b_j b_i) \wedge \beta_{I_1}(a_i) \text{ and}$$

$$D_{G_{P_1} * G_{P_2}} \gamma_I(a_j, b_j) = \sum \gamma_{I_1}(a_i a_k) \wedge \gamma_{I_2}(b_j) + \sum \gamma_{I_2}(b_j b_i) \wedge \gamma_{I_1}(a_i)$$

Therefore, from figure 6, degree of all vertices of  $G_{P_1} * G_{P_2}$  can be calculated as follows:

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(1,0) &= \alpha_{I_1}(1\ 0) \vee \alpha_{I_2}(0) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(1) + \alpha_{I_2}(0\ 2) \vee \alpha_{I_1}(1) \\ &= (0.9 \vee 0.7) + (0.8 \vee 0.8) + (0.9 \vee 0.8) \\ &= 0.9 + 0.8 + 0.9 = 2.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(1,0) &= \beta_{I_1}(1\ 0) \wedge \beta_{I_2}(0) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(1) + \beta_{I_2}(0\ 2) \wedge \beta_{I_1}(1) \\ &= (0.3 \wedge 0.4) + (0.4 \wedge 0.2) + (0.1 \wedge 0.4) \\ &= 0.3 + 0.2 + 0.1 = 0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(1,0) &= \gamma_{I_1}(1\ 0) \wedge \gamma_{I_2}(0) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(1) + \gamma_{I_2}(0\ 2) \wedge \gamma_{I_1}(1) \\ &= (0.2 \wedge 0.8) + (0.3 \wedge 0.1) + (0.7 \wedge 0.3) \\ &= 0.2 + 0.1 + 0.3 = 0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(0,0) &= \alpha_{I_1}(0\ 1) \vee \alpha_{I_2}(0) + \alpha_{I_1}(0\ 2) \vee \alpha_{I_2}(0) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(0) + \alpha_{I_2}(0\ 2) \vee \alpha_{I_1}(0) \\ &= (0.9 \vee 0.7) + (0.8 \vee 0.7) + (0.8 \vee 0.2) + (0.9 \vee 0.2) \\ &= 0.9 + 0.8 + 0.8 + 0.9 = 3.4 \end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \beta_I(0,0) &= \beta_{I_1}(0\ 1) \wedge \beta_{I_2}(0) + \beta_{I_1}(0\ 2) \wedge \beta_{I_2}(0) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(0) + \beta_{I_2}(0\ 2) \wedge \beta_{I_1}(0) \\
&= (0.3 \wedge 0.4) + (0.2 \wedge 0.4) (0.2 \wedge 0.6) + (0.1 \wedge 0.6) \\
&= 0.3 + 0.2 + 0.2 + 0.1 = 0.8
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \gamma_I(0,0) &= \gamma_{I_1}(0\ 1) \wedge \gamma_{I_2}(0) + \gamma_{I_1}(0\ 2) \wedge \gamma_{I_2}(0) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(0) + \gamma_{I_2}(0\ 2) \wedge \gamma_{I_1}(0) \\
&= (0.2 \wedge 0.8) + (0.1 \wedge 0.8) + (0.1 \wedge 0.4) + (0.7 \wedge 0.4) \\
&= 0.2 + 0.1 + 0.1 + 0.4 = 0.8
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \alpha_I(2,0) &= \alpha_{I_1}(2\ 0) \vee \alpha_{I_2}(0) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(2) + \alpha_{I_2}(0\ 2) \vee \alpha_{I_1}(2) \\
&= (0.8 \vee 0.7) + (0.8 \vee 0.3) + (0.9 \vee 0.3) \\
&= 0.8 + 0.8 + 0.9 = 2.5
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \beta_I(2,0) &= \beta_{I_1}(2\ 0) \wedge \beta_{I_2}(0) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(2) + \beta_{I_2}(0\ 2) \wedge \beta_{I_1}(2) \\
&= (0.2 \wedge 0.4) + (0.2 \wedge 0.5) + (0.1 \wedge 0.5) \\
&= 0.2 + 0.2 + 0.1 = 0.5
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \gamma_I(2,0) &= \gamma_{I_1}(2\ 0) \wedge \gamma_{I_2}(0) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(2) + \gamma_{I_2}(0\ 2) \wedge \gamma_{I_1}(2) \\
&= (0.1 \wedge 0.8) + (0.1 \wedge 0.7) + (0.7 \wedge 0.7) \\
&= 0.1 + 0.1 + 0.7 = 0.9
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \alpha_I(1,1) &= \alpha_{I_1}(1\ 0) \vee \alpha_{I_2}(1) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(1) + \alpha_{I_2}(1\ 2) \vee \alpha_{I_1}(1) \\
&= (0.9 \vee 0.5) + (0.8 \vee 0.8) + (0.7 \vee 0.8) \\
&= 0.9 + 0.8 + 0.9 = 2.6
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \beta_I(1,1) &= \beta_{I_1}(1\ 0) \wedge \beta_{I_2}(1) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(1) + \beta_{I_2}(1\ 2) \wedge \beta_{I_1}(1) \\
&= (0.3 \wedge 0.6) + (0.2 \wedge 0.4) + (0.2 \wedge 0.4) \\
&= 0.3 + 0.2 + 0.2 = 0.7
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \gamma_I(1,1) &= \gamma_{I_1}(1\ 0) \wedge \gamma_{I_2}(1) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(1) + \gamma_{I_2}(1\ 2) \wedge \gamma_{I_1}(1) \\
&= (0.2 \wedge 0.1) + (0.1 \wedge 0.3) + (0.1 \wedge 0.3) \\
&= 0.1 + 0.1 + 0.1 = 0.3
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \alpha_I(0,1) &= \alpha_{I_1}(0\ 2) \vee \alpha_{I_2}(1) + \alpha_{I_1}(0\ 1) \vee \alpha_{I_2}(1) + \alpha_{I_2}(0\ 1) \vee \alpha_{I_1}(0) + \alpha_{I_2}(1\ 2) \vee \alpha_{I_1}(0) \\
&= (0.8 \vee 0.5) + (0.9 \vee 0.5) + (0.8 \vee 0.2) + (0.7 \vee 0.2) \\
&= 0.8 + 0.9 + 0.8 + 0.7 = 3.2
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \beta_I(0,1) &= \beta_{I_1}(0\ 2) \wedge \beta_{I_2}(1) + \beta_{I_1}(0\ 1) \wedge \beta_{I_2}(1) + \beta_{I_2}(0\ 1) \wedge \beta_{I_1}(0) + \beta_{I_2}(1\ 2) \wedge \beta_{I_1}(0) \\
&= (0.2 \wedge 0.6) + (0.3 \wedge 0.6) (0.2 \wedge 0.6) + (0.2 \wedge 0.6) \\
&= 0.2 + 0.3 + 0.2 + 0.2 = 0.9
\end{aligned}$$

$$\begin{aligned}
D_{G_{P_1} * G_{P_2}} \gamma_I(0,1) &= \gamma_{I_1}(0\ 2) \wedge \gamma_{I_2}(1) + \gamma_{I_1}(0\ 1) \wedge \gamma_{I_2}(1) + \gamma_{I_2}(0\ 1) \wedge \gamma_{I_1}(0) + \gamma_{I_2}(1\ 2) \wedge \gamma_{I_1}(0) \\
&= (0.1 \wedge 0.1) + (0.2 \wedge 0.1) + (0.1 \wedge 0.4) + (0.1 \wedge 0.4)
\end{aligned}$$

$$= 0.1+0.1+0.1+0.1=0.4$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(2,1) &= \alpha_{I_1}(0 \ 2) \vee \alpha_{I_2}(1) + \alpha_{I_2}(0 \ 1) \vee \alpha_{I_1}(2) + \alpha_{I_2}(1 \ 2) \vee \alpha_{I_1}(2) \\ &= (0.8 \vee 0.5) + (0.8 \vee 0.3) + (0.7 \vee 0.3) \\ &= 0.8+0.8+0.7=2.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(2,1) &= \beta_{I_1}(0 \ 2) \wedge \beta_{I_2}(1) + \beta_{I_2}(0 \ 1) \wedge \beta_{I_1}(2) + \beta_{I_2}(1 \ 2) \wedge \beta_{I_1}(2) \\ &= (0.2 \wedge 0.6) + (0.2 \wedge 0.5) + (0.2 \wedge 0.5) \\ &= 0.2+0.2+0.2=0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(2,1) &= \gamma_{I_1}(0 \ 2) \wedge \gamma_{I_2}(1) + \gamma_{I_2}(0 \ 1) \wedge \gamma_{I_1}(2) + \gamma_{I_2}(1 \ 2) \wedge \gamma_{I_1}(2) \\ &= (0.1 \wedge 0.1) + (0.1 \wedge 0.7) + (0.1 \wedge 0.7) \\ &= 0.1+0.1+0.1=0.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(1,2) &= \alpha_{I_1}(0 \ 1) \vee \alpha_{I_2}(2) + \alpha_{I_2}(0 \ 2) \vee \alpha_{I_1}(1) + \alpha_{I_2}(1 \ 2) \vee \alpha_{I_1}(1) \\ &= (0.9 \vee 0.4) + (0.9 \vee 0.8) + (0.7 \vee 0.8) \\ &= 0.9+0.9+0.8=2.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(1,2) &= \beta_{I_1}(0 \ 1) \wedge \beta_{I_2}(2) + \beta_{I_2}(0 \ 2) \wedge \beta_{I_1}(1) + \beta_{I_2}(1 \ 2) \wedge \beta_{I_1}(1) \\ &= (0.3 \wedge 0.3) + (0.1 \wedge 0.4) + (0.2 \wedge 0.4) \\ &= 0.3+0.1+0.2=0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(1,2) &= \gamma_{I_1}(0 \ 1) \wedge \gamma_{I_2}(2) + \gamma_{I_2}(0 \ 2) \wedge \gamma_{I_1}(1) + \gamma_{I_2}(1 \ 2) \wedge \gamma_{I_1}(1) \\ &= (0.2 \wedge 0.9) + (0.7 \wedge 0.3) + (0.1 \wedge 0.3) \\ &= 0.2+0.3+0.1=0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(0,2) &= \alpha_{I_1}(0 \ 1) \vee \alpha_{I_2}(2) + \alpha_{I_1}(0 \ 2) \vee \alpha_{I_2}(2) + \alpha_{I_2}(1 \ 2) \vee \alpha_{I_1}(0) + \alpha_{I_2}(0 \ 2) \vee \alpha_{I_1}(0) \\ &= (0.9 \vee 0.4) + (0.8 \vee 0.4) + (0.7 \vee 0.2) + (0.9 \vee 0.2) \\ &= 0.9+0.8+0.7+0.9=3.3 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(0,2) &= \beta_{I_1}(0 \ 1) \wedge \beta_{I_2}(2) + \beta_{I_1}(0 \ 2) \wedge \beta_{I_2}(2) + \beta_{I_2}(1 \ 2) \wedge \beta_{I_1}(0) + \beta_{I_2}(0 \ 2) \wedge \beta_{I_1}(0) \\ &= (0.3 \wedge 0.3) + (0.2 \wedge 0.3) + (0.2 \wedge 0.6) + (0.1 \wedge 0.6) \\ &= 0.3+0.2+0.2+0.1=0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \gamma_I(0,2) &= \gamma_{I_1}(0 \ 1) \wedge \gamma_{I_2}(2) + \gamma_{I_1}(0 \ 2) \wedge \gamma_{I_2}(2) + \gamma_{I_2}(1 \ 2) \wedge \gamma_{I_1}(0) + \gamma_{I_2}(0 \ 2) \wedge \gamma_{I_1}(0) \\ &= (0.2 \wedge 0.9) + (0.1 \wedge 0.9) + (0.1 \wedge 0.4) + (0.7 \wedge 0.4) \\ &= 0.2+0.1+0.1+0.4=0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \alpha_I(2,2) &= \alpha_{I_1}(0 \ 2) \vee \alpha_{I_2}(2) + \alpha_{I_2}(1 \ 2) \vee \alpha_{I_1}(2) + \alpha_{I_2}(0 \ 2) \vee \alpha_{I_1}(2) \\ &= (0.8 \vee 0.4) + (0.7 \vee 0.3) + (0.9 \vee 0.3) \\ &= 0.8+0.7+0.9=2.4 \end{aligned}$$

$$\begin{aligned} D_{G_{P_1} * G_{P_2}} \beta_I(2,2) &= \beta_{I_1}(0 \ 2) \wedge \beta_{I_2}(2) + \beta_{I_2}(1 \ 2) \wedge \beta_{I_1}(2) + \beta_{I_2}(0 \ 2) \wedge \beta_{I_1}(2) \\ &= (0.2 \wedge 0.3) + (0.2 \wedge 0.5) + (0.1 \wedge 0.5) \end{aligned}$$

$$= 0.2+0.2+0.1=0.5$$

$$\begin{aligned} D_{G_{P_1 * G_{P_2}}} \gamma_I(2,2) &= \gamma_{I_1}(0,2) \wedge \gamma_{I_2}(2) + \gamma_{I_2}(1,2) \wedge \gamma_{I_1}(2) + \gamma_{I_2}(0,2) \wedge \gamma_{I_1}(2) \\ &= (0.1 \wedge 0.9) + (0.1 \wedge 0.7) + (0.7 \wedge 0.7) \\ &= 0.1+0.1+0.7=0.9 \end{aligned}$$

### 3.7 Definition

A Strong Pythagorean neutrosophic fuzzy graph (SPNFG) is  $G_{SP} = (V, E)$  with  $\alpha_I, \beta_I,$  and  $\gamma_I$  from  $V$  to  $[0,1]$  such that  $0 \leq \alpha_I^2(a) + \beta_I^2(a) + \gamma_I^2(a) \leq 2$  for all  $a \in V$  and  $E \subseteq V \times V$  with  $\alpha_I, \beta_I,$  and  $\gamma_I$  from  $V \times V$  to  $[0,1]$  such that

- $\alpha_I(ab) = \max\{\alpha_I(a), \alpha_I(b)\}$
- $\beta_I(ab) = \min\{\beta_I(a), \beta_I(b)\}$
- $\gamma_I(ab) = \min\{\gamma_I(a), \gamma_I(b)\}$  with  $0 \leq \alpha_I^2(ab) + \beta_I^2(ab) + \gamma_I^2(ab) \leq 2$  for all  $ab \in E_I$

Then  $G_{SP} = (V, E)$  is epitomized as  $G_{SP}(V_I, E_I, \alpha_I, \beta_I, \gamma_I)$  and is stated to as the graph of the SPNF ideal  $I_P(\alpha_I, \beta_I, \gamma_I)$ .

### 3.8 Theorem

Maximal product of two strong graphs of PNFIMFGNR is also a strong graph of PNFIMFGNR.

#### Proof

Let  $G_{SP_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{SP_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  be two strong graph structures of PNFIMFG. Then, their edges exist by the definition of maximal product and has its membership, indeterminacy and non-membership values as follows:

$$\begin{aligned} \alpha_I((a_1, b_1)(a_2, b_2)) &= \{ \alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1, b_2) \text{ where } a_1 = a_2 \ \& \ b_1, b_2 \in E_{I_2} \\ &\quad \alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_1, a_2) \text{ where } b_1 = b_2 \ \& \ a_1, a_2 \in E_{I_1} \}, \\ \beta_I((a_1, b_1)(a_2, b_2)) &= \{ \beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1, b_2) \text{ where } a_1 = a_2 \ \& \ b_1, b_2 \in E_{I_2} \\ &\quad \beta_{I_2}(b_2) \wedge \beta_{I_1}(a_1, a_2) \text{ where } b_1 = b_2 \ \& \ a_1, a_2 \in E_{I_1} \} \text{ and} \\ \gamma_I((a_1, b_1)(a_2, b_2)) &= \{ \gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1, b_2) \text{ where } a_1 = a_2 \ \& \ b_1, b_2 \in E_{I_2} \\ &\quad \gamma_{I_2}(b_2) \wedge \gamma_{I_1}(a_1, a_2) \text{ where } b_1 = b_2 \ \& \ a_1, a_2 \in E_{I_1} \} \text{ such that} \end{aligned}$$

$$0 \leq [\alpha_I((a_1, b_1)(a_2, b_2))]^2 + [\beta_I((a_1, b_1)(a_2, b_2))]^2 + [\gamma_I((a_1, b_1)(a_2, b_2))]^2 \leq 2$$

Let  $G_{SP_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{SP_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  be two graphs of the ideals  $I_1$  and  $I_2$  and are strong graphs of PNFIMFG then,  $G_{SP_1} * G_{SP_2} = (V_I, E_I, \alpha_I, \beta_I, \gamma_I)$  is the maximal product of two strong graphs of PNFIMFG is given by

$$V_I = V_{I_1} \times V_{I_2} \text{ and } E_I = \{(a_1, b_1)(a_2, b_2) / a_1 = a_2, b_1, b_2 \in E_{I_2} \text{ or } b_1 = b_2 \ \& \ a_1, a_2 \in E_{I_1}\} \text{ with}$$

$\alpha_I(a, b) = \alpha_{I_1}(a) \vee \alpha_{I_2}(b)$  for all  $(a, b) \in V_I = V_{I_1} \times V_{I_2}$  as membership value,  
 $\beta_I(a, b) = \beta_{I_1}(a) \wedge \beta_{I_2}(b)$  for all  $(a, b) \in V_I = V_{I_1} \times V_{I_2}$  as indeterminacy value and  
 $\gamma_I(a, b) = \gamma_{I_1}(a) \wedge \gamma_{I_2}(b)$  for all  $(a, b) \in V_I = V_{I_1} \times V_{I_2}$  as non-membership value of the vertices. Then, by the definition, edges of maximal product consisting of membership, indeterminacy and non-membership values are:

$$\alpha_I((a_1, b_1) (a_2, b_2)) = \{ \alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \ \& \ b_1 \ b_2 \in E_{I_2}$$

$$\alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \ \& \ a_1 \ a_2 \in E_{I_1} \},$$

$$\beta_I((a_1, b_1) (a_2, b_2)) = \{ \beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \ \& \ b_1 \ b_2 \in E_{I_2}$$

$$\beta_{I_2}(b_2) \wedge \beta_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \ \& \ a_1 \ a_2 \in E_{I_1} \} \text{ and}$$

$$\gamma_I((a_1, b_1) (a_2, b_2)) = \{ \gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1 b_2) \text{ where } a_1 = a_2 \ \& \ b_1 \ b_2 \in E_{I_2}$$

$$\gamma_{I_2}(b_2) \wedge \gamma_{I_1}(a_1 a_2) \text{ where } b_1 = b_2 \ \& \ a_1 \ a_2 \in E_{I_1} \} \text{ such that}$$

$$0 \leq [\alpha_I((a_1, b_1) (a_2, b_2))]^2 + [\beta_I((a_1, b_1) (a_2, b_2))]^2 + [\gamma_I((a_1, b_1) (a_2, b_2))]^2 \leq 2$$

Hence, it satisfies the definition of strong graph of PNFIM $\Gamma$ GNR.

### 3.9 Result

The maximal product is a strong graph of PNFIM $\Gamma$ GNR, and it is independent of their products. Maximal product  $G_{SP} = G_{SP_1} * G_{SP_2}$  may be of a strong PNFIM $\Gamma$ GNR when  $G_{SP_1}$  and  $G_{SP_2}$  are not strong graphs of PNFIM $\Gamma$ GNR.

#### Case (i)

$a_1 = a_2$  and  $b_1 \ b_2 \in E_{I_2}$  Therefore,

$$\alpha_I((a_1, b_1) (a_2, b_2)) = \alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1 \ b_2)$$

$$= \alpha_{I_1}(a_1) \vee [\alpha_{I_2}(b_1) \wedge \alpha_{I_2}(b_2)]$$

$$= [\alpha_{I_1}(a_1) \vee \alpha_{I_2}(b_1)] \wedge [\alpha_{I_1}(a_2) \vee \alpha_{I_2}(b_2)]$$

$$= \alpha_I(a_1 b_1) \wedge \alpha_I(a_2 b_2)$$

#### Case (ii)

$b_1 = b_2$  and  $a_1 \ a_2 \in E_{I_1}$

Therefore,

$$\alpha_I((a_1, b_1) (a_2, b_2)) = \alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_1 \ a_2)$$

$$= \alpha_{I_2}(b_2) \vee [\alpha_{I_1}(a_1) \wedge \alpha_{I_1}(a_2)]$$

$$= [\alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_1)] \wedge [\alpha_{I_2}(b_2) \vee \alpha_{I_1}(a_2)]$$

$$= \alpha_I(a_1 b_1) \wedge \alpha_I(a_2 b_2)$$

For all the edges of the maximal product.

Similarly, we find indeterminacy and non-membership values for the edges as follows:

**Case (i)**

$a_1 = a_2$  and  $b_1, b_2 \in E_{I_2}$ . Then,

$$\begin{aligned} \beta_I((a_1, b_1) (a_2, b_2)) &= \beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1, b_2) \\ &= \beta_{I_1}(a_1) \wedge [\beta_{I_2}(b_1) \vee \beta_{I_2}(b_2)] \\ &= [\beta_{I_1}(a_1) \wedge \beta_{I_2}(b_1)] \vee [\beta_{I_1}(a_1) \wedge \beta_{I_2}(b_2)] \\ &= \beta_I(a_1, b_1) \vee \beta_I(a_1, b_2) \end{aligned}$$

**Case (ii)**

$b_1 = b_2$  and  $a_1, a_2 \in E_{I_1}$ . Then

$$\begin{aligned} \beta_I((a_1, b_1) (a_2, b_2)) &= \beta_{I_2}(b_1, b_2) \wedge \beta_{I_1}(a_1, a_2) \\ &= \beta_{I_2}(b_1, b_2) \wedge [\beta_{I_1}(a_1) \vee \beta_{I_1}(a_2)] \\ &= [\beta_{I_2}(b_1, b_2) \wedge \beta_{I_1}(a_1)] \vee [\beta_{I_2}(b_1, b_2) \wedge \beta_{I_1}(a_2)] \\ &= \beta_I(a_1, b_1) \vee \beta_I(a_2, b_1) \text{ for every maximal product edge.} \end{aligned}$$

**Case (i)**

$a_1 = a_2$  and  $b_1, b_2 \in E_{I_2}$ . Therefore,

$$\begin{aligned} \gamma_I((a_1, b_1) (a_2, b_2)) &= \gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1, b_2) \\ &= \gamma_{I_1}(a_1) \wedge [\gamma_{I_2}(b_1) \vee \gamma_{I_2}(b_2)] \\ &= [\gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_1)] \vee [\gamma_{I_1}(a_1) \wedge \gamma_{I_2}(b_2)] \\ &= \gamma_I(a_1, b_1) \vee \gamma_I(a_1, b_2) \end{aligned}$$

**Case (ii)**

$b_1 = b_2$  and  $a_1, a_2 \in E_{I_1}$ . Therefore,

$$\begin{aligned} \gamma_I((a_1, b_1) (a_2, b_2)) &= \gamma_{I_2}(b_1, b_2) \wedge \gamma_{I_1}(a_1, a_2) \\ &= \gamma_{I_2}(b_1, b_2) \wedge [\gamma_{I_1}(a_1) \vee \gamma_{I_1}(a_2)] \\ &= [\gamma_{I_2}(b_1, b_2) \wedge \gamma_{I_1}(a_1)] \vee [\gamma_{I_2}(b_1, b_2) \wedge \gamma_{I_1}(a_2)] \\ &= \gamma_I(a_1, b_1) \vee \gamma_I(a_2, b_1) \end{aligned}$$

For all the edges of the maximal product such that

$$0 \leq [\alpha_I((a_1, b_1) (a_2, b_2))]^2 + [\beta_I((a_1, b_1) (a_2, b_2))]^2 + [\gamma_I((a_1, b_1) (a_2, b_2))]^2 \leq 2$$

Hence  $G_{SP} = G_{SP_1} * G_{SP_2}$  is a strong graph of PNFIMΓGNR when  $G_{SP_1}$  and  $G_{SP_2}$  are not strong graphs of NFIMΓGNR

**3.10 Theorem**

If  $G_{SP_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{SP_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  are the graphs of strong

PNFIMΓGNR such that  $\alpha_{I_1}(a_i) \leq \alpha_{I_2}(a_i b_j), \beta_{I_1}(a_i) \geq \beta_{I_2}(a_i b_j), \gamma_{I_1}(a_i) \geq \gamma_{I_2}(a_i b_j)$

therefore, the vertex degree of maximal product  $G_{SP_1} * G_{SP_2}((V_I, E_I, \alpha_I, \beta_I, \gamma_I))$  is given by

$$D_{G_{SP_1} * G_{SP_2}} \alpha_I(a_i, b_j) = D_{G_{SP_1}} \alpha_{I_1}(a_i) \alpha_{I_2}(b_j) + D_{G_{SP_2}} \alpha_{I_2}(b_j)$$

$$D_{G_{SP_1} * G_{SP_2}} \beta_I(a_i, b_j) = D_{G_{SP_1}} * \beta_{I_1}(a_i) \beta_{I_2}(b_j) + D_{G_{SP_2}} \beta_{I_2}(b_j)$$

$$D_{G_{SP_1} * G_{SP_2}} \gamma_I(a_i, b_j) = D_{G_{SP_1}} * \gamma_{I_1}(a_i) \gamma_{I_2}(b_j) + D_{G_{SP_2}} \gamma_{I_2}(b_j)$$

**Proof**

Let  $G_{SP_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  and  $G_{SP_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  are the graphs of strong PNFIMΓGNR such that

$$\alpha_{I_1}(a_i) \leq \alpha_{I_2}(b_i b_j) \text{ then } \alpha_{I_1}(a_i b_j) \leq \alpha_{I_2}(b_j),$$

$$\beta_{I_1}(a_i) \geq \beta_{I_2}(b_i b_j) \text{ then } \beta_{I_1}(a_i b_j) \geq \beta_{I_2}(b_j) \text{ and}$$

$$\gamma_{I_1}(a_i) \geq \gamma_{I_2}(b_i b_j) \text{ then } \gamma_{I_1}(a_i b_j) \geq \gamma_{I_2}(b_j) \text{ for } a_i \in V_{I_1}, a_i a_j \in E_{I_1}, b_i \in V_{I_2}, b_i b_j \in E_{I_2}.$$

Therefore, the vertex degree of  $G_{SP_1} * G_{SP_2}$  maximal product are:

$$D_{G_{SP_1} * G_{SP_2}} \alpha_I(a_i, b_j) = \sum \alpha_{I_1}(a_i a_j) \vee \alpha_{I_2}(b_j) + \sum \alpha_{I_2}(b_i b_j) \vee \alpha_{I_1}(a_i)$$

$$D_{G_{SP_1} * G_{SP_2}} \beta_I(a_i, b_j) = \sum \beta_{I_1}(a_i a_j) \wedge \beta_{I_2}(b_j) + \sum \beta_{I_2}(b_i b_j) \wedge \beta_{I_1}(a_i) \text{ and}$$

$$D_{G_{SP_1} * G_{SP_2}} \gamma_I(a_i, b_j) = \sum \gamma_{I_1}(a_i a_j) \wedge \gamma_{I_2}(b_j) + \sum \gamma_{I_2}(b_i b_j) \wedge \gamma_{I_1}(a_i)$$

$$\Rightarrow D_{G_{SP_1} * G_{SP_2}} \alpha_I(a_i, b_j) = \sum \alpha_{I_2}(b_j) + \sum \alpha_{I_2}(b_i b_j)$$

$$D_{G_{SP_1} * G_{SP_2}} \beta_I(a_i, b_j) = \sum \beta_{I_2}(b_j) + \sum \beta_{I_2}(b_i b_j) \text{ and}$$

$$D_{G_{SP_1} * G_{SP_2}} \gamma_I(a_i, b_j) = \sum \gamma_{I_2}(b_j) + \sum \gamma_{I_2}(b_i b_j)$$

$$\Rightarrow D_{G_{SP_1} * G_{SP_2}} \alpha_I(a_i, b_j) = D_{G_{SP_1}} * \alpha_{I_1}(a_i) \alpha_{I_2}(b_j) + D_{G_{SP_2}} \alpha_{I_2}(b_j)$$

$$D_{G_{SP_1} * G_{SP_2}} \beta_I(a_i, b_j) = D_{G_{SP_1}} * \beta_{I_1}(a_i) \beta_{I_2}(b_j) + D_{G_{SP_2}} \beta_{I_2}(b_j) \text{ and}$$

$$D_{G_{SP_1} * G_{SP_2}} \gamma_I(a_i, b_j) = D_{G_{SP_1}} * \gamma_{I_1}(a_i) \gamma_{I_2}(b_j) + D_{G_{SP_2}} \gamma_{I_2}(b_j)$$

**3.11 Example**

Consider  $G_{SP_1}(V_{I_1}, E_{I_1}, \alpha_{I_1}, \beta_{I_1}, \gamma_{I_1})$  for  $I_1 = \{0,2\}$  of  $Z_4$  and  $G_{SP_2}(V_{I_2}, E_{I_2}, \alpha_{I_2}, \beta_{I_2}, \gamma_{I_2})$  for  $I_2 = \{0\}$  of  $Z_4$  therefore,  $G_{SP} = G_{SP_1} * G_{SP_2}$  is a maximal product of  $G_{SP_1}$  and  $G_{SP_2}$ . This is explained in the following example.

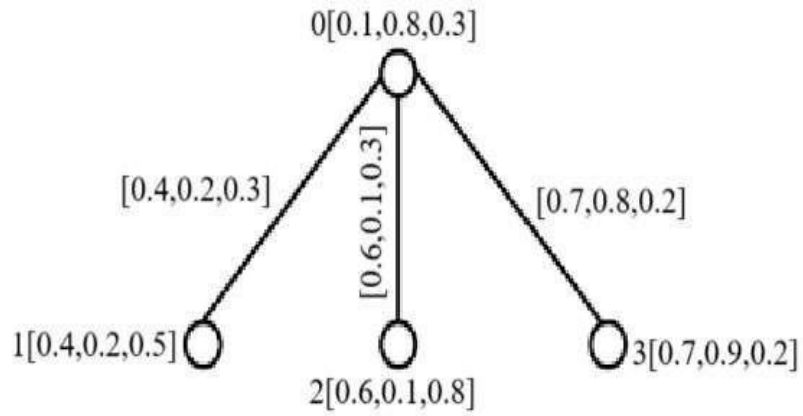


Figure 7: Graph  $G_{SP_1}$

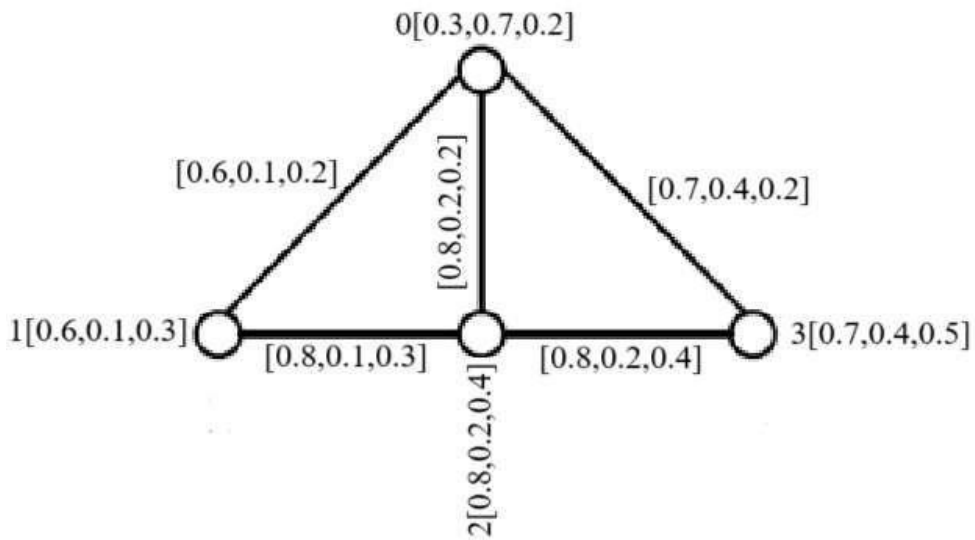


Figure 8: Graph  $G_{SP_2}$

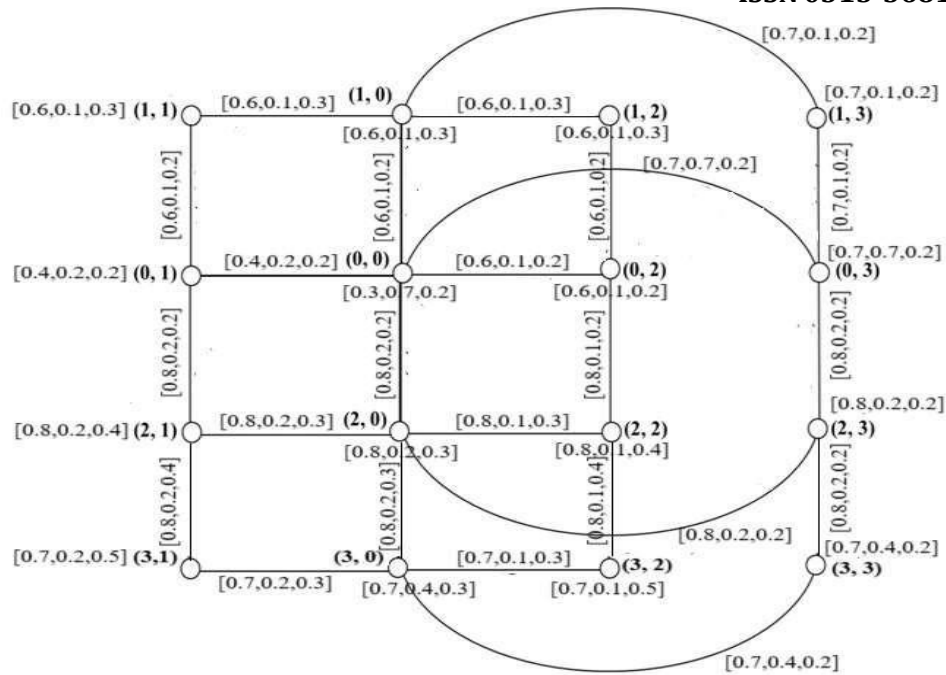


Figure 9 : Graph of  $G_{SP_1} * G_{SP_2}$

The  $\alpha_I$  value for  $G_{SP_1} * G_{SP_2}$  by theorem is obtained as follows.

By theorem,

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(0,0) &= D_{G_{SP_1}} * (0)\alpha_{I_2}(0) + D_{G_{SP_2}} \alpha_{I_2}(0) \\ &= 3(0.1) + (0.4 + 0.6 + 0.7) \\ &= 2.0 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(0,1) &= D_{G_{SP_1}} * (0)\alpha_{I_2}(1) + D_{G_{SP_2}} \alpha_{I_2}(1) \\ &= 3(0.4) + (0.4) \\ &= 1.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(0,2) &= D_{G_{SP_1}} * (0)\alpha_{I_2}(2) + D_{G_{SP_2}} \alpha_{I_2}(2) \\ &= 3(0.6) + (0.6) \\ &= 2.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(0,3) &= D_{G_{SP_1}} * (0)\alpha_{I_2}(3) + D_{G_{SP_2}} \alpha_{I_2}(3) \\ &= 3(0.7) + (0.7) \\ &= 2.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(1,0) &= D_{G_{SP_1}} * (1)\alpha_{I_2}(0) + D_{G_{SP_2}} \alpha_{I_2}(0) \\ &= 2(0.1) + (0.4 + 0.6 + 0.7) \\ &= 1.9 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(1,1) &= D_{G_{SP_1}} * (1)\alpha_{I_2}(1) + D_{G_{SP_2}} \alpha_{I_2}(1) \\ &= 2(0.4) + (0.4) \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(1,2) &= D_{G_{SP_1}} * (1)\alpha_{I_2}(2) + D_{G_{SP_2}} \alpha_{I_2}(2) \\ &= 2(0.6) + (0.6) \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(1,3) &= D_{G_{SP_1}} * (1)\alpha_{I_2}(3) + D_{G_{SP_2}} \alpha_{I_2}(3) \\ &= 2(0.7) + (0.7) \\ &= 2.1 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(2,0) &= D_{G_{SP_1}} * (2)\alpha_{I_2}(0) + D_{G_{SP_2}} \alpha_{I_2}(0) \\ &= 3(0.1) + (0.4 + 0.6 + 0.7) \\ &= 2.0 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(2,1) &= D_{G_{SP_1}} * (2)\alpha_{I_2}(1) + D_{G_{SP_2}} \alpha_{I_2}(1) \\ &= 3(0.4) + (0.4) \\ &= 1.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(2,2) &= D_{G_{SP_1}} * (2)\alpha_{I_2}(2) + D_{G_{SP_2}} \alpha_{I_2}(2) \\ &= 3(0.6) + (0.6) \\ &= 2.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(2,3) &= D_{G_{SP_1}} * (2)\alpha_{I_2}(3) + D_{G_{SP_2}} \alpha_{I_2}(3) \\ &= 3(0.7) + (0.7) \\ &= 2.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(3,0) &= D_{G_{SP_1}} * (3)\alpha_{I_2}(0) + D_{G_{SP_2}} \alpha_{I_2}(0) \\ &= 2(0.1) + (0.4 + 0.6 + 0.7) \\ &= 1.9 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(3,1) &= D_{G_{SP_1}} * (3)\alpha_{I_2}(1) + D_{G_{SP_2}} \alpha_{I_2}(1) \\ &= 2(0.4) + (0.4) \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(3,2) &= D_{G_{SP_1}} * (3)\alpha_{I_2}(2) + D_{G_{SP_2}} \alpha_{I_2}(2) \\ &= 2(0.6) + (0.6) \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \alpha_I(3,3) &= D_{G_{SP_1}} * (3)\alpha_{I_2}(3) + D_{G_{SP_2}} \alpha_{I_2}(3) \\ &= 2(0.7) + (0.7) \\ &= 2.1 \end{aligned}$$

Similarly, the  $\beta_I$  value for  $G_{SP_1} * G_{SP_2}$  by theorem is obtained as follows.

By theorem,

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(0,0) &= D_{G_{SP_1}} * (0)\beta_{I_2}(0) + D_{G_{SP_2}} \beta_{I_2}(0) \\ &= 3(0.8) + (0.2 + 0.1 + 0.8) \end{aligned}$$

$$= 3.5$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(0,1) &= D_{G_{SP_1}} * (0)\beta_{I_2}(1) + D_{G_{SP_2}} \beta_{I_2}(1) \\ &= 3(0.2) + (0.2) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(0,2) &= D_{G_{SP_1}} * (0)\beta_{I_2}(2) + D_{G_{SP_2}} \beta_{I_2}(2) \\ &= 3(0.1) + (0.1) \\ &= 0.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(0,3) &= D_{G_{SP_1}} * (0)\beta_{I_2}(3) + D_{G_{SP_2}} \beta_{I_2}(3) \\ &= 3(0.9) + (0.8) \\ &= 3.5 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(1,0) &= D_{G_{SP_1}} * (1)\beta_{I_2}(0) + D_{G_{SP_2}} \beta_{I_2}(0) \\ &= 2(0.8) + (0.2 + 0.1 + 0.8) \\ &= 2.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(1,1) &= D_{G_{SP_1}} * (1)\beta_{I_2}(1) + D_{G_{SP_2}} \beta_{I_2}(1) \\ &= 2(0.2) + (0.2) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(1,2) &= D_{G_{SP_1}} * (1)\beta_{I_2}(2) + D_{G_{SP_2}} \beta_{I_2}(2) \\ &= 2(0.1) + (0.1) \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(1,3) &= D_{G_{SP_1}} * (1)\beta_{I_2}(3) + D_{G_{SP_2}} \beta_{I_2}(3) \\ &= 2(0.9) + (0.8) \\ &= 2.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(2,0) &= D_{G_{SP_1}} * (2)\beta_{I_2}(0) + D_{G_{SP_2}} \beta_{I_2}(0) \\ &= 3(0.8) + (0.2 + 0.1 + 0.8) \\ &= 3.5 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(2,1) &= D_{G_{SP_1}} * (2)\beta_{I_2}(1) + D_{G_{SP_2}} \beta_{I_2}(1) \\ &= 3(0.2) + (0.2) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(2,2) &= D_{G_{SP_1}} * (2)\beta_{I_2}(2) + D_{G_{SP_2}} \beta_{I_2}(2) \\ &= 3(0.1) + (0.1) \\ &= 0.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(2,3) &= D_{G_{SP_1}} * (2)\beta_{I_2}(3) + D_{G_{SP_2}} \beta_{I_2}(3) \\ &= 3(0.9) + (0.8) \\ &= 3.5 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(3,0) &= D_{G_{SP_1}} * (3)\beta_{I_2}(0) + D_{G_{SP_2}} \beta_{I_2}(0) \\ &= 2(0.8) + (0.2 + 0.1 + 0.8) \\ &= 2.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(3,1) &= D_{G_{SP_1}} * (3)\beta_{I_2}(1) + D_{G_{SP_2}} \beta_{I_2}(1) \\ &= 2(0.2) + (0.2) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(3,2) &= D_{G_{SP_1}} * (3)\beta_{I_2}(2) + D_{G_{SP_2}} \beta_{I_2}(2) \\ &= 2(0.1) + (0.1) \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \beta_I(3,3) &= D_{G_{SP_1}} * (3)\beta_{I_2}(3) + D_{G_{SP_2}} \beta_{I_2}(3) \\ &= 2(0.9) + (0.8) \\ &= 2.6 \end{aligned}$$

Similarly, the  $\gamma_I$  value for  $G_{SP_1} * G_{SP_2}$  by theorem is obtained as follows.

By theorem,

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(0,0) &= D_{G_{SP_1}} * (0)\gamma_{I_2}(0) + D_{G_{SP_2}} \gamma_{I_2}(0) \\ &= 3(0.3) + (0.3 + 0.3 + 0.2) \\ &= 1.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(0,1) &= D_{G_{SP_1}} * (0)\gamma_{I_2}(1) + D_{G_{SP_2}} \gamma_{I_2}(1) \\ &= 3(0.5) + (0.3) \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(0,2) &= D_{G_{SP_1}} * (0)\gamma_{I_2}(2) + D_{G_{SP_2}} \gamma_{I_2}(2) \\ &= 3(0.8) + (0.3) \\ &= 2.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(0,3) &= D_{G_{SP_1}} * (0)\gamma_{I_2}(3) + D_{G_{SP_2}} \gamma_{I_2}(3) \\ &= 3(0.2) + (0.2) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(1,0) &= D_{G_{SP_1}} * (1)\gamma_{I_2}(0) + D_{G_{SP_2}} \gamma_{I_2}(0) \\ &= 2(0.3) + (0.3 + 0.3 + 0.2) \\ &= 1.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(1,1) &= D_{G_{SP_1}} * (1)\gamma_{I_2}(1) + D_{G_{SP_2}} \gamma_{I_2}(1) \\ &= 2(0.5) + (0.3) \\ &= 1.3 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(1,2) &= D_{G_{SP_1}} * (1)\gamma_{I_2}(2) + D_{G_{SP_2}} \gamma_{I_2}(2) \\ &= 2(0.8) + (0.3) \end{aligned}$$

$$= 1.9$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(1,3) &= D_{G_{SP_1}} * (1) \gamma_{I_2}(3) + D_{G_{SP_2}} \gamma_{I_2}(3) \\ &= 2(0.2) + (0.2) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(2,0) &= D_{G_{SP_1}} * (2) \gamma_{I_2}(0) + D_{G_{SP_2}} \gamma_{I_2}(0) \\ &= 3(0.3) + (0.3 + 0.3 + 0.2) \\ &= 1.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(2,1) &= D_{G_{SP_1}} * (2) \gamma_{I_2}(1) + D_{G_{SP_2}} \gamma_{I_2}(1) \\ &= 3(0.5) + (0.3) \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(2,2) &= D_{G_{SP_1}} * (2) \gamma_{I_2}(2) + D_{G_{SP_2}} \gamma_{I_2}(2) \\ &= 3(0.8) + (0.3) \\ &= 2.7 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(2,3) &= D_{G_{SP_1}} * (2) \gamma_{I_2}(3) + D_{G_{SP_2}} \gamma_{I_2}(3) \\ &= 3(0.2) + (0.2) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(3,0) &= D_{G_{SP_1}} * (3) \gamma_{I_2}(0) + D_{G_{SP_2}} \gamma_{I_2}(0) \\ &= 2(0.3) + (0.3 + 0.3 + 0.2) \\ &= 1.4 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(3,1) &= D_{G_{SP_1}} * (3) \gamma_{I_2}(1) + D_{G_{SP_2}} \gamma_{I_2}(1) \\ &= 2(0.5) + (0.3) \\ &= 1.3 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(3,2) &= D_{G_{SP_1}} * (3) \gamma_{I_2}(2) + D_{G_{SP_2}} \gamma_{I_2}(2) \\ &= 2(0.8) + (0.3) \\ &= 1.9 \end{aligned}$$

$$\begin{aligned} D_{G_{SP_1} * G_{SP_2}} \gamma_I(3,3) &= D_{G_{SP_1}} * (3) \gamma_{I_2}(3) + D_{G_{SP_2}} \gamma_{I_2}(3) \\ &= 2(0.2) + (0.2) \\ &= 0.6 \end{aligned}$$

#### 4. CONCLUSION

The notion of the maximal product of the graph of a PNF ideal and strong graph of a PNF ideal of the  $M\Gamma$  group in near rings has been presented, along with examples that illustrate its properties. Given that graphs are used in a wide range of fields, including networking, logistics, and the maximal product of PNF graphs, other uses for graphs may be investigated in the future.

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