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LAPLACIAN MINIMUM COVERING EXTENDED ENERGY OF A GRAPH

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Abstract:

In the current research, we present the idea of the Laplacian minimum covering extended e´nergy of a graph $E_C(G)$ and calculate the Laplacian minimum covering extended energies of a complete graph, star graph, crown graph, cocktail party graph, and full bipartite graph.

Keywords: Laplacian minimum covering extended eigen values, Laplacian Minimum covering extended matrix.

Introduction

The conceptualization of a graph's energy becomes first mounted by using I. Gutman [7] in 1978. Allow G be (η,μ) graph. Permit $A = (a_{ij})$ be the graph's adjacency matrix. The graph G's eigen values are the non-incremental eigen values ($\lambda_1, \lambda_2, \lambda_\eta$) of A . G's eigen values are actual and general, by considering that A is actual symmetric. Concerning the mathematical aspects of the theory of graph energy, see critiques [8], papers [1, 2, 4, 5, 6, 10, 16], and the references listed therein; more studies on dominating energy can be observed in [3, 15].

Extended Energy

Consider G a simple graph, with the edge set E , node set $V = \{v_1, v_2, .., v_\eta\}$ of order n and G's extended matrix is the $\eta \times \eta$ matrix stated as $A_{ex}(G) := (a_{ij})$, Where

$$(a_{ij}) = \begin{Bmatrix} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right) & if\ v_i v_j \epsilon E(G) \\ 0 & otherwise \end{Bmatrix}$$

$f_n(G, \lambda) = \det(\lambda I - A_{ex}(G))$ indicates the characteristic polynomial of $A_{ex}(G)$. The eigen values of $A_{ex}(G)$ are real numbers as it is symmetric and real.

For the graph G, extended energy is outlined as $E_{ex}(G) := \sum_{i=1}^{n} |\lambda_i|$.

Laplacian energy

For a graph G, the laplacian energy was installed in 2006 via I. Gutman and B. Zhou [9]. Permit μ and η represents nodes, edges of a graph G . The graph a η × η matrix, its laplacian minimum covering extended energy of G is given by $L_{ex}^{C}(G) := (x_{ij})$, [11, 12] Where $x_{ij} = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } j = i \text{ and } v_i \epsilon D \\ -\frac{1}{2}\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right) & \text{if } v_j v_i \epsilon E(G) \end{cases}$

Results and Discussions

Properties of Laplacian minimum covering extended eigen values

Theorem 2.1. *Assume that the node set of a simple graph G is* $V = \{v_1, v_2, .., v_\eta\}$ *, the edge set is E, and C={ $u_1, u_2, ..., u^\mu$ }, indicates minimum covering set. Upon denoting the eigenvalues of the Laplacian minimum covering extended matrix* $L_{ex}^{C}(G)$ as $\ddot{\xi}_i, \ddot{\xi}_2, \dots, \ddot{\xi}_\eta$

We obtain $(i) \sum_{i=1}^{\eta} \ddot{\xi}_i = 2|E| - |C|$

$(ii) \sum_{i=1}^{\eta} \ddot{\xi}_i^{\,2} = \sum_{i=1}^{\eta} (-c_i + d_i)^2 + \frac{1}{2} \sum_{j>1} \left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right)^2$

Where $c_i = \begin{cases} 1 & if\ v_i \in C \\ 0 & otherwise. \end{cases}$

Proof: i) We are aware that the trace of $L_{ex}^{c}(G)$ is equal to the sum of eigenvalus of $L_{ex}^{c}(G)$

$$\therefore \sum_{i=1}^{\eta} \ddot{\xi}_i = \sum_{i=1}^{\eta} a_{ii} = -|C| + \sum_{i=1}^{\eta} d_i$$

$$= -|C| + 2|E| = -k + 2\mu$$

ii) Similarly the sum of squares of the eigen values of $L_{ex}^{c}(G)$ is the trace of $[L_{ex}^{c}(G)\ ]^2$.

$$\therefore \sum_{i=1}^{\eta} \ddot{\xi}_i^{\,2} = \sum_{j=1}^{\eta} \sum_{i=1}^{\eta} a_{ji}\, a_{ij}$$

$$= \sum_{j \neq 1} a_{ji}\, a_{ij} + \sum_{i=1}^{\eta} (a_{ii})^2$$

$$= 2 \sum_{j>i} \left(a_{ij}\right)^2 + \sum_{i=1}^{\eta} (a_{ii})^2$$

$$= 2 \sum_{j>i} \left(\frac{1}{2}\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right)\right)^2 + \sum_{i=1}^{\eta} (-c_i + d_i)^2,$$

Where $c_i = \begin{cases} 1 & if\ v_i \in C \\ 0 & otherwise. \end{cases}$

$$= \frac{1}{2} \sum_{j>i} \left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right)^2 + \sum_{i=1}^{\eta} (-c_i + d_i)^2$$

Theorem 2. 2. *If $L_{ex}^{c}(G)$, the Laplacian minimum covering extended matrix, has a rational sum of its absolute eigen values*

$$\sum_{i=1} |\ddot{\xi}| \equiv |C| \, (mod2).$$

Proof: Let $\ddot{\xi}_1, \ddot{\xi}_2, \dots, \ddot{\xi}_\eta$ refrers to eigen values of Laplacian minimum covering extended matrix $L_{ex}^{c}(G)$ of a graph G, of which $\ddot{\xi}_i, \ddot{\xi}_2, \dots, \ddot{\xi}_r$ are non negative and the rest are non-positive, then

$$\sum_{i=1}^{\eta} |\ddot{\xi}_i| = -(\ddot{\xi}_{r+1} + \cdots + \ddot{\xi}_\eta) + (\ddot{\xi}_i + \ddot{\xi}_2 + \cdots + \ddot{\xi}_r)$$

$$\sum_{i=1}^{\eta} |\ddot{\xi}_i| = -(\ddot{\xi}_1 + \ddot{\xi}_2 + \cdots + \ddot{\xi}_\eta) + 2(\ddot{\xi}_i + \ddot{\xi}_2 + \cdots + \ddot{\xi}_r)$$

$$= -\sum_{i=1}^{\eta} \ddot{\xi}_i + 2(\ddot{\xi}_i + \ddot{\xi}_2 + \cdots + \ddot{\xi}_r)$$

$$= -(2|E| - |C|) + 2(\ddot{\xi}_i + \ddot{\xi}_2 + \cdots + \ddot{\xi}_r)$$

$$= |C| + 2(\ddot{\xi}_i + \ddot{\xi}_2 + \cdots + \ddot{\xi}_r - |E|)$$

$$\sum_{i=1} |\ddot{\xi}| \equiv |C| \, (\mathrm{mod}2).$$

The aforementioned outcome is comparable to [3]'s Parity Theorem 3.7.

Theorem 2.3. *Assume that there are η nodes and μ edges in graph G. We find that $E_{ex}^{C}(G) \in (-2\mu + 2t, 2\mu + 2t)$, where t represents an integer such that $\sum_{i=1}^{\eta} |\ddot{\xi}_1| \equiv |C| (mod2)$,is the rational number resulting from the sum of the absolute Laplacian minimum covering extended eigen values.*

Proof:

$$\sum_{i=1}^{\eta} \left|\ddot{\xi}_1 - \frac{2\mu}{\eta}\right| - 2\mu + \sum_{i=1}^{\eta} |\ddot{\xi}_1|$$

$$E_{ex}^{C}(G) \leq 2\mu + \sum_{i=1}^{\eta} |\ddot{\xi}_1|$$

$$2\mu + 2t$$

$$E_{ex}^{C}(G) \geq 2\mu + \sum_{i=1}^{\eta} |\ddot{\xi}_1|$$

$$-2\mu + 2t$$

i.e., $E_{ex}^{C}(G) \in (-2\mu + 2t, 2\mu + 2t)$

Bounds for Laplacian minimum covering extended energy

Upper as well as lower bounds on a graph's ordinary *e´nergy* were provided by McClelland [13].
The following theorem yields similar constraints for $E_{ex}^{C}(G)$.

Theorem 3.1. *(Upper bound) For a simple graph G with μ edges and η nodes then*

$$E_{ex}^{C}(G) \leq +\sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2\right)}$$

Proof: Cauchy Schwarz inequality is

$$\left(\sum_{i=1}^{\eta} b_i^2\right)\left(\sum_{i=1}^{\eta} a_i^2\right) \geq \left(\sum_{i=1}^{\eta} b_i a_i\right)^2$$

If $b_i = |\xi_i|$ and $a_i = 1$ then

$$\left(\sum_{i=1}^{\eta} \ddot{\xi}_i^2\right)\left(\sum_{i=1}^{\eta} 1\right) \geq \left(\sum_{i=1}^{\eta}|\ddot{\xi}_i|\right)^2$$

$$\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2\right) \geq \left(\sum_{i=1}^{\eta}|\ddot{\xi}_i|\right)^2$$

$$\sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2\right)} \geq \sum_{i=1}^{\eta}\ddot{\xi}_i$$

By Triangle in equality

$$\left|\ddot{\xi}_i-\frac{2\mu}{\eta}\right| \leq \left|\ddot{\xi}_i\right| + \left|\frac{2\mu}{\eta}\right| \quad \forall\, i = 1,2,\ldots,\eta$$

i.e.,

$$\left|\ddot{\xi}_i-\frac{2\mu}{\eta}\right| \leq \left|\ddot{\xi}_i\right| + \frac{2\mu}{\eta} \quad \forall\, i$$

$$\sum_{i=1}^{\eta}\left|\ddot{\xi}_i-\frac{2\mu}{\eta}\right| \leq \sum_{i=1}^{\eta}|\ddot{\xi}_i| + \sum_{i=1}^{\eta}\frac{2\mu}{\eta}$$

i.e., $\sum_{i=1}^{\eta}\left|\ddot{\xi}_i-\frac{2\mu}{\eta}\right| \leq \sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2\right)} + 2\mu$

$$\therefore\ ,\ \ E_{ex}^{C}(G) \leq \sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2\right)+2\mu}$$

Theorem3.2. *(Upper bound)For a simple graph G with η nodes and μ edges then*

$$E_{ex}^{C}(G) \leq \sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2-\frac{4\mu^2}{\eta}+\frac{\mu}{\eta}\right)}$$

Proof: Cauchy Schwarz inequality is

$$\left(\sum_{i=1}^{\eta} b_i^2\right)\left(\sum_{i=1}^{\eta} a_i^2\right) \geq \left(\sum_{i=1}^{\eta} b_i a_i\right)^2$$

If $b_i = \left|-\frac{2\mu}{\eta}+\ddot{\xi}_i\right|$ and $a_i = 1$ then

$$\left(\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota-\frac{2\mu}{\eta}\right|^2\right)\left(\sum_{i=1}^{\eta}1\right) \geq \left(\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota-\frac{2\mu}{\eta}\right|\right)^2$$

i.e..,

$$\eta\left(\sum_{i=1}^{\eta}\xi_i^2+\sum_{i=1}^{\eta}\frac{4\mu^2}{\eta^2}-\frac{4\mu}{\eta}\sum_{i=1}^{\eta}\ddot{\xi}_\iota\right) \geq [E_{ex}^{C}(G)]^2$$

$$= \eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2+\frac{4\mu^2\eta}{\eta^2}-\frac{4\mu}{\eta}(2\eta-k)\right)$$

$$= \eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2+\frac{4\mu^2}{\eta}-\frac{8\mu^2}{\eta}+\frac{4\mu k}{\eta}\right)$$

=

$$\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2-\frac{4\mu^2}{\eta}+\frac{4\mu k}{\eta}\right)$$

Therefore

$$\sqrt{\eta\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2-\frac{4\mu^2}{\eta}+\frac{\mu}{\eta}\right)} \geq E_{ex}^{C}(G)$$

Theorem 3.3 *(Lower bound) For a simple graph G with η nodes and μ edges and*

$P = |L_{ex}^{C}(G)|$ *then* $(E_{ex}^{C}(G) \geq \sqrt{\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 + \eta(-1+\eta)P^{\frac{2}{\eta}}\right)} - 2\mu$

Proof: Consider

$$\left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right)^2$$

$$= \left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right) * \left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right)$$

$$\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|^2 + \sum_{j\neq i}|\ddot{\xi}_j||\ddot{\xi}_\iota|$$

$\therefore \sum_{j\neq i}|\ddot{\xi}_j||\ddot{\xi}_\iota| = \left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right)^2 - \sum_{i=1}^{\eta}|\ddot{\xi}_\iota|^2$ .. (3.3.1)

For $\eta(-1+\eta)$ terms applying inequality between the geometric and arithmetic means, we have

$$\frac{\sum_{j\neq i}|\ddot{\xi}_j||\ddot{\xi}_\iota|}{\eta(-1+\eta)} \geq \left[\prod_{i\neq j}|\ddot{\xi}_j||\ddot{\xi}_\iota|\right]^{\frac{1}{\eta(-1+\eta)}}$$

i.e..,

$$\left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right)^2 - \sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 \geq \eta(-1+\eta)\left[\prod_{i=1}^{\eta}|\ddot{\xi}_\iota|\right]^{\frac{2}{\eta}}$$

$$\left(\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\right)^2 \geq \sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 \geq \eta(-1+\eta)\left[\prod_{i=1}^{\eta}|\ddot{\xi}_\iota|\right]^{\frac{2}{\eta}}$$

$$\therefore \sum_{i=1}^{\eta}|\ddot{\xi}_\iota| \geq \sqrt{\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 + \eta(-1+\eta)P^{\frac{2}{\eta}}\right)}$$

We know that

$$|\ddot{\xi}_\iota| - \left|\frac{2\mu}{\eta}\right| \leq \left|\ddot{\xi}_\iota - \frac{2\mu}{\eta}\right| \quad \forall\, i$$

$$\text{i.e., } |\ddot{\xi}_\iota| - \frac{2\mu}{\eta} \leq \left|\ddot{\xi}_\iota - \frac{2\mu}{\eta}\right| \quad \forall\, i$$

$$\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right| - \sum_{i=1}^{\eta}\frac{2\mu}{\eta} \le \sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota - \frac{2\mu}{\eta}\right|$$

i.e., $\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right| - 2\mu \le E_{ex}^{C}(G)$

$$E_{ex}^{C}(G) \ge \sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right| - 2\mu$$

$$\ge \sqrt{\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 + \eta(-1+\eta)P^{\frac{2}{\eta}}\right)} - 2\mu$$

$$\sum_{j\neq i}\left|\ddot{\xi}_j\right|\left|\ddot{\xi}_\iota\right| \ge \eta(-1+\eta)\left[\prod_{i\neq j}\left|\ddot{\xi}_j\right|\left|\ddot{\xi}_\iota\right|\right]^{\frac{1}{\eta(-1+\eta)}}$$

From 3.3.1 we get

$$\left(\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right|\right)^2 - \sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right|^2 \ge \eta(-1+\eta)\left[\prod_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right|^{2(-1+\eta)}\right]^{\frac{1}{\eta(-1+\eta)}}$$

$$\therefore E_{ex}^{C}(G) \ge \sqrt{\left(\sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 + \eta(-1+\eta)P^{\frac{2}{\eta}}\right)} - 2\mu$$

Theorem 3.4 *For a graph G with μ edges and η nodes .Let* $\left|\ddot{\xi}_\iota\right| \ge \left|\ddot{\xi}_\iota\right| \ge \cdots \ge \left|\ddot{\xi}_\iota\right| > 0$ *be a non Incremental order of eigen values of* $L_{ex}^{C}(G)$ *then*

$$E_{ex}^{C}(G) \ge \frac{\sum_{i=1}^{\eta}(-c_i+d_i)^2 + \frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2 + \eta\left|\ddot{\xi}_J\right|\left|\ddot{\xi}_\eta\right|}{\left(\left|\ddot{\xi}_J\right| + \left|\ddot{\xi}_\eta\right|\right)} - 2\mu$$

Proof : The inequality $ra_i \le b_i \le Ra_i$ holds if $a_i \neq 0$, R, r and b_i are real numbers Let $a_i = 0$,b_i, r and R be real numbers satisfying $ra_i \le b_i \le Ra_i$,then the following inequality holds [Theorem 2, [14]].

$$\sum_{i=1}^{\eta} b_i^2 + rR\sum_{i=1}^{\eta} a_i \le (r+R)\sum_{i=1}^{\eta} a_i b_i$$

Put $b_i = \left|\ddot{\xi}_\iota\right|, a_i = 1, r = \left|\ddot{\xi}_\eta\right|$ and $R = \left|\ddot{\xi}_\iota\right|$ then

$$\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right|^2 + \left|\ddot{\xi}_\iota\right|\left|\ddot{\xi}_\eta\right|\sum_{i=1}^{\eta} 1 \le \left(\left|\ddot{\xi}_\iota\right| + \left|\ddot{\xi}_\eta\right|\right)\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota\right|$$

i.e..,

$$\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2+\eta|\ddot{\xi}_j||\ddot{\xi}_\eta|\leq(|\ddot{\xi}_\iota|+|\ddot{\xi}_\eta|)\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|$$

$$\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|\geq\frac{\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2+\eta|\ddot{\xi}_j||\ddot{\xi}_\eta|}{(|\ddot{\xi}_j|+|\ddot{\xi}_\eta|)}$$

We know that

$$E_{ex}^{C}(G)=\sum_{i=1}^{\eta}\left|\ddot{\xi}_\iota-\frac{2\mu}{\eta}\right|$$

$$E_{ex}^{C}(G)\geq\sum_{i=1}^{\eta}|\ddot{\xi}_\iota|-\left|\frac{2\mu}{\eta}\right|$$

$$E_{ex}^{C}(G)\quad\geq\frac{\sum_{i=1}^{\eta}(-c_i+d_i)^2+\frac{1}{2}\sum_{j>1}\left(\frac{d_j}{d_i}+\frac{d_i}{d_j}\right)^2+\eta|\ddot{\xi}_j||\ddot{\xi}_\eta|}{(|\ddot{\xi}_j|+|\ddot{\xi}_\eta|)}-2\mu$$

LAPLACIAN MINIMUM COVERING EXTENDED ENERGY OF SOME STANDARD GRAPHS

Theorem 4.1 *For the star graph, laplacian minimum covering extended energy is given by*

$$K_{1,1-\eta}\ is\ (\eta-2)+\frac{\sqrt{\eta^5-4\eta^4+4\eta^3+6\eta^2-12\eta+5}}{-1+\eta}$$

where $v_0\geq 2$.

Proof. Consider a star graph $K_{1,-1+\eta}$ with vertex set $V=\{v_0,v_1,v_2,\dots,v_{-1+\eta}\}$ and minimum covering set $C=\{v_0\}$

$$A_{ex}^{C}(K_{-1+\eta})=\begin{pmatrix}1 & \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & \dots & \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right)\\ \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 0 & 0 & \dots & 0\\ \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 0 & 0 & \dots & 0\end{pmatrix}_{\eta\times\eta}$$

and

$$C(K_{1,-1+\eta})=\begin{pmatrix} -1+\eta & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & .. \\ 0 & 0 & 0 & & 1 \end{pmatrix}_{\eta\times\eta}$$

$$\therefore L_{ex}^{C}(K_{1,-1+\eta}) = D(K_{1,-1+\eta}) - A_{ex}^{C}(K_{1,-1+\eta})$$

$$L_{ex}^{C}(K_{1,-1+\eta}) =$$

$$\begin{pmatrix} \eta-2 & -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & \ldots & -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) \\ -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 1 & 0 & & 0 \\ -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 0 & 1 & & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ --\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & 0 & 0 & & 1 \end{pmatrix}$$

Laplacian minimum covering extended spectrum is

$$Spec\left(L_{ex}^{C}(K_{1,-1+\eta})\right)=$$

$$\begin{pmatrix} 1 & \frac{(\eta^2-2\eta+1)+\sqrt{\eta^5-4\eta^4+4\eta^3+6\eta^2-12\eta+5}}{2\eta-2} & \frac{(\eta^2-2\eta+1)+\sqrt{\eta^5-4\eta^4+4\eta^3+6\eta^2-12\eta+5}}{2\eta-2} \\ \eta-2 & 1 & 1 \end{pmatrix}$$

Average degree $=\frac{2\mu}{\eta}=\frac{2(-1+\eta)}{\eta}$

Hence Laplacian minimum covering extended energy,

Theorem 4.2. *For the complete graph, laplacian minimum covering extended energy is given by*

$$(\eta^2-3\eta+2)+\sqrt{\eta^2+2\eta-3}\,, where\ \eta \geq 2.$$

Proof. Consider a star graph K_η with vertex set V=$\{v_1, v_2, \ldots, v_\eta\}$ and minimum covering set $C=\{v_1\}$

$$E_{ex}^{C}(K_{1,1-\eta}) = (\eta-2) + \frac{\sqrt{\eta^5-4\eta^4+4\eta^3+6\eta^2-12\eta+5}}{-1+\eta}$$

$$A_{ex}^{C}(K_\eta)=\begin{pmatrix} 1 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 1 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ \frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 1 & \dots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ . & . & . & . & . \\ . & . & . & . & . \\ .. & . & . & . & . \\ \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & 0 \end{pmatrix}_{\eta\times\eta}$$

$$D(K_\eta)=\begin{pmatrix} -1+\eta & 0 & 0 & \dots & 0 \\ 0 & -1+\eta & 0 & \dots & 0 \\ 0 & 0 & -1+\eta & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & -1+\eta \end{pmatrix}_{\eta\times\eta}$$

$$\therefore L_{ex}^{C}(K_\eta)=D(K_\eta)-A_{ex}^{C}(K_\eta)$$

$$L_{ex}^{C}(K_\eta)=\begin{pmatrix} \eta-2 & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \eta-2 & \frac{-1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ -\frac{1}{2}\left(\frac{-1+\eta}{1}+\frac{1}{-1+\eta}\right) & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \eta-2 & \dots & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right)- \\ . & . & . & . & . \\ . & . & . & . & . \\ .. & . & . & & . \\ -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & -\frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \dots & -1+\eta \end{pmatrix}_{\eta\times\eta}$$

Minimum covering extended spectrum is

$$Spec\left(L_{ex}^{C}(K_\eta)\right)=$$

$$\begin{pmatrix} -1+\eta & \frac{(\eta-1)+\sqrt{\eta^2+2\eta-3}}{2} & \frac{(\eta-1)-\sqrt{\eta^2+2\eta-3}}{2} \\ \eta-2 & 1 & 1 \end{pmatrix}$$

Average degree $=\frac{2\mu}{\eta}=2\frac{\frac{\eta(-1+\eta)}{2}}{\eta}=-1+\eta$

Hence Laplacian minimum covering extended energy,

$$E_{ex}^{C}(K_\eta)=(\eta^2-3\eta+2)+\sqrt{\eta^2+2\eta-3}$$

Theorem 4.3. *For the crown graph, laplacian minimum covering extended energy is given by* $(4\eta^2 - 8\eta + 5) + \sqrt{8\eta^2 - 24\eta + 13}$, *where* $\eta \geq 2$.

Proof. *Consider the crown graph* ${S_{2\eta}}^0$ *with vertex set* $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$, *the minimum covering set*

$$\left(\begin{array}{ccccccccccc} & u_1 & u_2 & u_3 & \cdots & u_n & v_1 & v_2 & v_3 & \cdots & v_n \\ u_1 & 1 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ u_2 & 0 & 1 & 0 & \cdots & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & 0 \\ v_1 & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & 0 & 0 & \cdots & 0 \\ v_2 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_n & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & \cdots & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

$$D(S_{2n}^0) = \left(\begin{array}{ccccccccccc} & u_1 & u_2 & u_3 & \cdots & u_n & v_1 & v_2 & v_3 & \cdots & v_n \\ u_1 & -1+\eta & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ u_2 & 0 & -1+\eta & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & 0 & 0 & 0 & \cdots & -1+\eta & 0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & 0 & 0 & \cdots & 0 & -1+\eta & 0 & 0 & \cdots & 0 \\ v_2 & 0 & 0 & 0 & \cdots & 0 & 0 & -1+\eta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

$$L_{ex}^c(S_{2\eta}^0) = D(S_\eta^0) - A_{ex}^c(S_{2\eta}^0) =$$

$$\left(\begin{array}{ccccccccccc} & u_1 & u_2 & u_3 & \cdots & u_\eta & v_1 & v_2 & v_3 & \cdots & v_n \\ u_1 & -2+\eta & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & -1 & \cdots & -1 \\ u_2 & 0 & -2+\eta & 0 & \cdots & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & -1 & \cdots & -1 \\ u_3 & 0 & 0 & -2+\eta & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_\eta & 0 & 0 & 0 & \cdots & -2+\eta & -1 & -1 & -1 & \cdots & 0 \\ v_1 & 0 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & -1 & \cdots & -1 & -2+\eta & 0 & 0 & \cdots & 0 \\ v_2 & \frac{1}{2}\left(\frac{-1+\eta}{-1+\eta}+\frac{-1+\eta}{-1+\eta}\right) & 0 & -1 & \cdots & -1 & 0 & -1+\eta & 0 & \cdots & 0 \\ v_3 & -1 & -1 & 0 & \cdots & -1 & 0 & 0 & -1+\eta & \cdots & 0 \\ \vdots & \vdots & -1 & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1+\eta \end{array}\right)$$

Laplacian minimum covering extended spectrum

$$Spec\left(L_{ex}^{C}\left(S_{2\eta}^{0}\right)\right)=\begin{pmatrix} \frac{(2\eta-3)+\sqrt{4\eta^2-8\eta+5}}{2} & \frac{(2\eta-3)-\sqrt{4\eta^2-8\eta+5}}{2} & \frac{(2\eta-3)+\sqrt{8\eta^2-24\eta+13}}{2} & \frac{(2\eta-3)+\sqrt{8\eta^2-24\eta+13}}{2} \\ 1 & 1 & 1 & \end{pmatrix}$$

Average degree $=\frac{2\mu}{\eta}=\frac{2\eta(-1+\eta)}{2\eta}=-1+\eta$

Laplacian minimum covering extended energy is $E_{ex}^{c}(S_{2n}^{0})=(4\eta^2-8\eta+5)+\sqrt{8\eta^2-24\eta+13}$

Theorem 4.5 *For the complete bi-partite graph, laplacian minimum covering extended energy is given by*

$$E_{ex}^{C}(K_{\mu,\eta})=(2\mu\eta-2\mu-\eta+1)+\frac{\sqrt{\mu\eta^5+\mu^2\eta^4-2\mu^2\eta^3+(\mu^4+2\mu^3+\mu^2)\eta^2+\mu^5\eta}}{\mu\eta}$$

Proof: For the complete bi partite graph $K_{\mu,\eta}(\mu\le\eta)$ with vertex set $V=\{u_1,u_2,\dots u_m,v_1,v_2,\dots v_n\}$. The Minimum covering set is $V=\{u_1,u_2,\dots,u_\mu\}$.

The Minimum covering extended matrix of complete bipartite graph is

$$A_{ex}^{C}(K_{\mu,\eta})=\begin{pmatrix} & v_1 & v_2 & \dots & v_\mu & u_1 & u_2 & \dots & u_\eta \\ v_1 & 1 & 0 & \dots & 0 & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ v_2 & 0 & 1 & \dots & 0 & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ . & . & . & . & . & & & & \\ . & . & . & . & . & & & & \\ v_\mu & 0 & 0 & \dots & 1 & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ u_1 & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \dots & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & 0 & \dots & 0 \\ u_2 & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{n}+\frac{\eta}{\mu}\right) & \dots & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & 0 & \dots & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ u_\eta & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \dots & \frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$D(K_{\mu,\eta})=\begin{pmatrix} & u_1 & u_2 & u_3 & \cdots & u_\eta & v_1 & v_2 & v_3 & \cdots & v_\mu \\ u_1 & \eta & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ u_2 & 0 & \eta & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_\eta & 0 & 0 & 0 & \cdots & \eta & 0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & 0 & 0 & \cdots & 0 & \mu & 0 & 0 & \cdots & 0 \\ v_2 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_\mu & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix}$$

$$L_{ex}^{c}(K_{\mu,\eta})=D(K_{\mu,\eta})-A_{ex}^{c}(K_{\mu,\eta})=$$

$$\begin{pmatrix} & v_1 & v_2 & v_3 & \cdots & v_\mu & u_1 & u_2 & u_3 & \cdots & u_\eta \\ v_1 & -1+\eta & 0 & 0 & \cdots & 0 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ v_2 & 0 & -1+\eta & 0 & \cdots & 0 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ v_3 & 0 & 0 & -1+\eta & \cdots & 0 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_\mu & 0 & 0 & 0 & \cdots & -1+\eta & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) \\ u_1 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \mu & 0 & 0 & \cdots & 0 \\ u_2 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & \mu & 0 & \cdots & 0 \\ u_3 & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_\eta & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & \cdots & -\frac{1}{2}\left(\frac{\mu}{\eta}+\frac{\eta}{\mu}\right) & 0 & 0 & 0 & \cdots & \mu \end{pmatrix}$$

$$Spec\left(L_{ex}^{C}\left(K_{\mu,\eta}\right)\right)=$$

$$\begin{pmatrix} -1+\eta & \mu & \frac{X+\sqrt{Y}}{2\mu\eta} & \frac{X-\sqrt{Y}}{2\mu\eta} \\ \mu-1 & -1+\eta & 1 & 1 \end{pmatrix}$$

where

$$X=\mu\eta^2+(\mu^2-\mu)\eta \qquad Y=\mu\eta^5+\mu^2\eta^4-2\mu^2\eta^3+(\mu^4+2\mu^3+\mu^2)\eta^2+\mu^5\eta$$

and

$$Average\deg ree=\frac{2\mu}{\eta}=\frac{2\mu\eta}{\mu+\eta}$$

Laplacian minimum covering extended energy is

$$E_{ex}^{C}(K_{\mu,\eta}) = (2\mu\eta - 2\mu - \eta + 1) + \frac{\sqrt{\mu\eta^5 + \mu^2\eta^4 - 2\mu^2\eta^3 + (\mu^4 + 2\mu^3 + \mu^2)\eta^2 + \mu^5\eta}}{\mu\eta}$$

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