b – Coloring of Extended Duplication Graph of Dragon graph($T_{n,1}$) and Sunlet graph(S_n) Networks

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Abstract

The b – chromatic number $\varphi(G)$, is the highest k for which G confess a b-coloring by k colors. A b-coloring of G by k colors is proper vertex k-coloring such that in each color class i there is a vertex x_i having neighbors in every other $k-1$ color classes. In this article we obtain b – chromatic number of (M[EDG($T_{n,1}$)]) middle graph of extended duplication graph of Dragon graph, middle graph of extended duplicate graph of Sunlet graph (M[EDG(S_n)]), and total graph of (T[EDG($T_{n,1}$)]) extended duplication graph of Dragon graph and total graph of extended duplicate graph of Sunlet graph T[EDG(S_n)].

***Keywords* -** b-coloing, middle graph, total graph, extended duplication graph, Dragon, Sunlet

1. Introduction

If no two neighbours in a graph G are given the same colour, then the graph is properly coloured [2]. In this article, "graph" refers to a finite, undirected, simple graph, while "colouring" refers to the colouring of vertex graphs.

If there is a vertex with color i that has at least one neighbor in each of the other color classes, then the proper k-coloring of a graph G is a b-coloring. A b-vertex is one such vertex. The *b-chromatic number* of G, denoted by $\varphi(G)$, is the biggest integer k for which G permits a b- coloring for k colors and G is known as *b-colorable graph*.

The idea of b-coloring was first proposed by Irving and Manlove [3], who investigated several results about it and demonstrated that calculating the b-chromatic number is an NP-hard problem [8].

The following upper bound has also been proved by Irving and Manlove [3]

$\varphi(G) \leq \Delta(G) + 1$ (1.1), where $\Delta(G)$ is the maximum degree of *G.*

Thirusangu et. al.[7] proposed the concept of extended duplication graph. A *duplication graph*[5] *of G* is symbolized by DG = (V_1, E_1) where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f : V \to V'$ is bijective (for v *in* V, we write $f(v) = v'$) and the edge set E_1 of DG is if and only if both $v_1 v_2'$ and $v_1' v_2$ are edges in E_1 then the edge $v_1 v_2$ is in E. The *extended duplication graph* of DG, denoted by EDG, is defined as, add an edge between any two vertex from V to any other vertex in V', except the terminal vertices of V and V'. For ease, we use $v_2$in V and v'_2 *in* V and so the edge $v_2 v'_2$is formed.

By adding a new vertex to each of a graph G's edges and connecting those pairs of new vertices with edges that recline on neighbouring edges of G[4,6], one may obtain the middle graph[1] of a graph G, which is represented by M(G).

T(G), a new graph whose vertex set is the union of vertex and edge sets of G, represents the total graph[1] of a graph G. Two vertices of T(G) are neighbours if they are either two adjacent vertices, two adjacent edges, or an incident vertex with an edge of G[4,6].

2. b-chromatic number of middle graph of extended duplication graph of Dragon graph

A cycle graph C_n and a singleton graph K_1 are connected by a bridge to form a graph known as a Dragon graph ($T_{n,1}$). The extended duplication graph of Dragon graph is denoted by EDG($T_{n,1}$).

Theorem 2.1

The b-chromatic number $\varphi(M[EDG(T_{n,1})])$ of middle graph of extended duplication graph of Dragon graph is six for $n \geq 4$

Proof:

Let the middle graph of extended duplication graph of Dragon $T_{n,1}$ is denoted by M[EDG($T_{n,1}$)] with 4n+5 vertices and 6n+14 edges with maximum degree 6 and minimum degree 1. It contains 2n-4 vertices each of degree 2 and degree 4, degree 1 of two vertices, 4 vertices each of degree 3 and degree 5. Three vertices with degree 6.

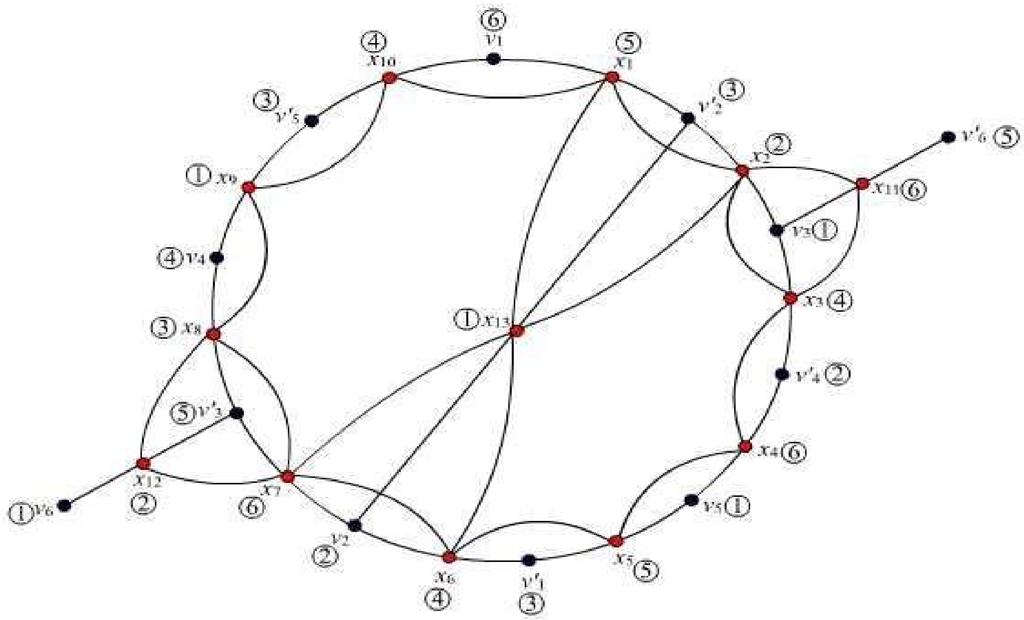

Figure 1 $\boldsymbol{M[EDG(T_{5,1})]}$

As $\Delta(M[EDG(T_{n,1})]) = 6$ by (1.1), $\varphi(M[EDG(T_{n,1})]) \leq 7$. If $\varphi(M[EDG(T_{n,1})]) \leq 7$ then $M[EDG(T_{n,1})]$ must have seven vertices of degree 6, which is impossible, as $M[EDG(T_{n,1})]$ has only three vertices with degree 6. So, $\varphi(M[EDG(T_{n,1})]) \neq 7$.

Due to adjacency of vertices in M$[EDG(T_{n,1})]$ maximum six b- vertices can be produed for any proper coloring. Consider the 6-coloring (c_1, c_2, c_3, c_4, c_5, c_6) of M$[EDG(T_{n,1})]$, allot the color c_1 to x_{2n+3} , allot the color c_2 for x_2 and v_2 , allot the color c_3 for v'_2 , x_{n+3} $when\ n\ is\ odd\ and$ v'_2, x_{n+4} $when\ n\ is\ even$. Allot the color c_4 to x_{n+1} $when\ n\ is\ odd\ and$ x_{n+2} $when\ n\ is\ even$, allot the color c_5 to x_1 , allot the color c_6 to x_{n+2} $when\ n\ is\ odd\ and$ x_{n+3} $when\ n\ is\ even$ and for the remaining vertices allot one of the allowed color - such color exists.

The b - vertices x_{2n+3} , x_2 , x_{n+3}, x_{n+1} , x_1 , x_{n+2} for the color classes ($c_1, c_2, c_3, c_4,$ c_5 and c_6) respectively produces the b - chromatic coloring when n ≡ 1(mod 2) and the b - vertices x_{2n+3}, x_2 , x_{n+4}, x_{n+2}, x_1 , x_{n+3} for the color classes $c_1, c_2, c_3, c_4,$ c_5 and c_6 respectively produces the b - chromatic coloring when n ≡ 0(mod 2).

Hence $\varphi(M[EDG(T_{n,1})]) = 6, n \geq 4$

3. b-chromatic number of middle graph of extended duplicate graph of sunlet graph

A n-sunlet graph S_n is a graph on 2n vertices obtained by appending n pendent edges to a cycle graph S_n . The extended duplicate graph of sunlet graph is denoted by EDG(S_n) with 2n vertices and 2n+1 edges.

Theorem 3.1

The b-chromatic number of middle graph of extended duplicate graph of sunlet graph is $\varphi(M[EDG(S_n)]) = 7, n \geq 5$.

Proof:

Let M[EDG(S_n)] be the middle graph of the extended duplicate graph of sunlet S_n, which has 8n+1 vertices and 14n+8 edges. It contains 2n vertices of degree 1, 2n-1 vertices of degree 3, 2n-1 vertices of degree 4, two vertices of degree 5, 2n-4 vertices of degree 6, four vertices of degree 7, and one vertex of degree 8.

As $\Delta(M[EDG(S_n)]) = 8$ by (1.1), $\varphi(M[EDG(S_n)]) \leq 9$. If $\varphi(M[EDG(S_n)]) \leq 9$ then $M[EDG(S_n)]$ must have nine vertices of degree 8, which is not possible, as $M[EDG(S_n)]$ has only one vertex of degree 8. Consequently, $\varphi(M[EDG(S_n)]) \neq 9$. If $\varphi(M[EDG(S_n)]) = 8$ then $M[EDG(S_n)]$ must have eight vertices of degree 7, which is also not possible, as $M[EDG(S_n)]$ has only four vertices of degree 7. Consequently, $\varphi(M[EDG(S_n)]) \neq 8$.

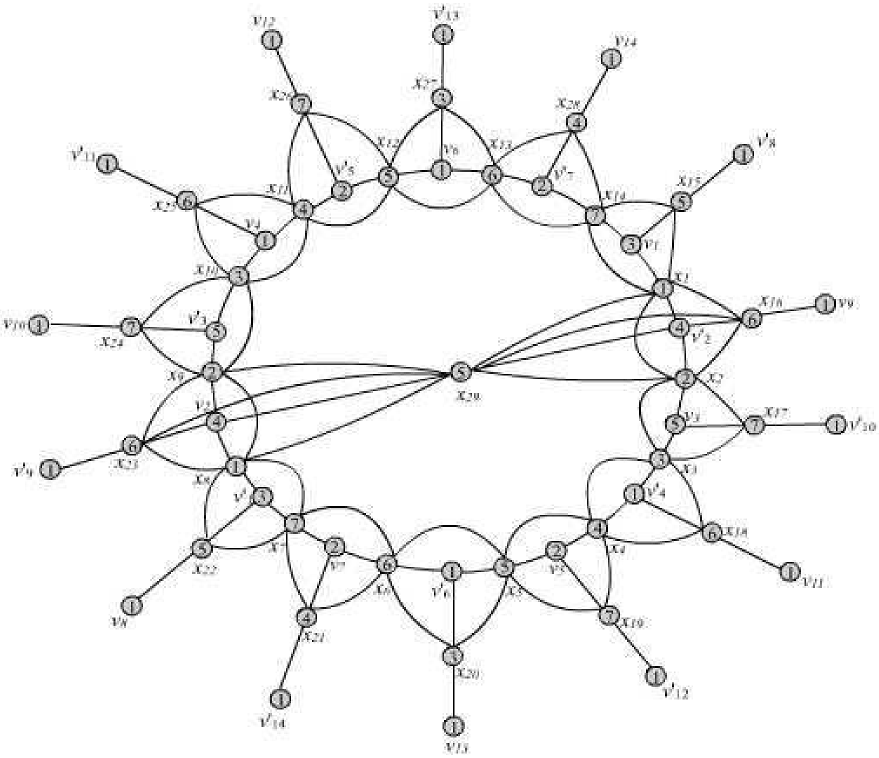

Figure 2 $\boldsymbol{M[EDG(S_7)]}$

Due to adjacency of vertices in $M[EDG(S_n)]$ at most seven b- vertices can be generated for any proper coloring.Consider the following 7-coloring ($c_1, c_2, c_3, c_4, c_5, c_6, c_7$) of $M[EDG(S_n)]$, An easy check shows that this is a b - coloring with b - vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7$for the color classes $c_1, c_2, c_3, c_4, c_5, c_6$ *and* c_7 respectively.

Hence $\varphi(M[EDG(S_n)]) = 7, n \geq 5$.

4. b-chromatic number of total graph of extended duplication graph of Dragon graph

Theorem 4.1

The b-chromatic number $(\varphi(T[EDG(T_{n,1})]))$ of total graph of extended duplication graph of Dragon graph is $Seven\ for\ n \geq 4$

Proof:

Let the total graph of extended duplication graph of Dragon $T_{n,1}$ is symbolized by $T[EDG(T_{n,1})]$ with 4n+5 vertices and 8n+17 edges with maximum degree 6 and minimum degree 2. it contains 4n-8 vertices of degree 4, two vertices with degree 2, 4 vertices with degree 5 and seven vertices with degree 6.

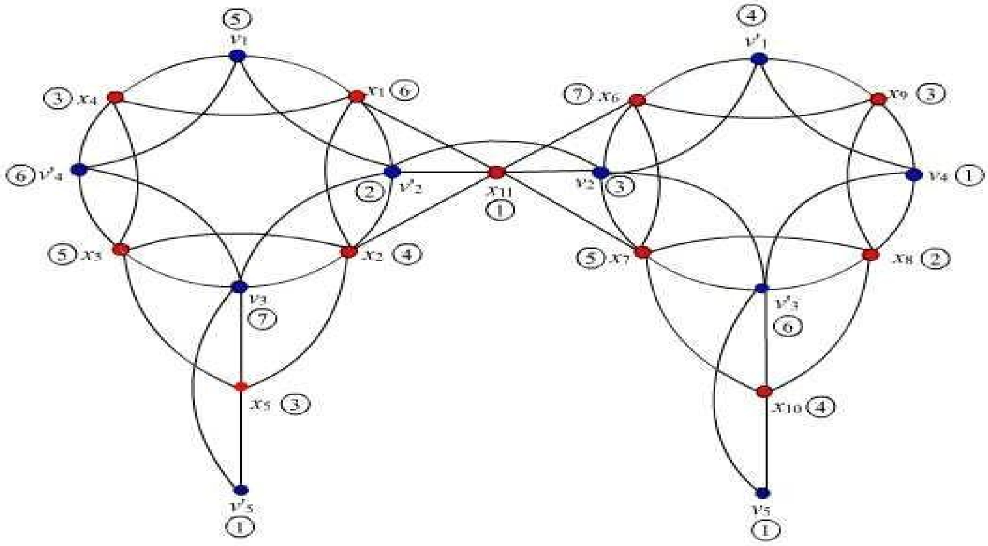

Figure 3 $\boldsymbol{T}[\boldsymbol{EDG}(\boldsymbol{T}_{4,1})]$

As $\Delta(T[EDG(T_{n,1})]) = 6$ by (1.1), $\varphi(T[EDG(T_{n,1})]) \leq 7$. If $\varphi(T[EDG(T_{n,1})]) \leq 7$ then $T[EDG(T_{n,1})]$ must contain seven vertices with degree 6, which is true.

Due to adjacency of vertices in $T[EDG(T_{n,1})]$ maximum seven b- vertices can be produced for any proper coloring. Consider the 7-coloring ($c_1, c_2, c_3, c_4, c_5, c_6, c_7$) of $T[EDG(T_{n,1})]$, allot the color c_1 to x_{2n+3} , allot the color c_2 for v_2' , allot the color c_3 for v_2 allot the color c_4 to x_2 ,allot the color c_5 to x_{n+3} *when n is even and* x_{n+2} *when n is odd*, allot the color c_6 to v'_3, allot the color c_7 to v_3 and for the remaining vertices allot any of the allowed color – there is such color.

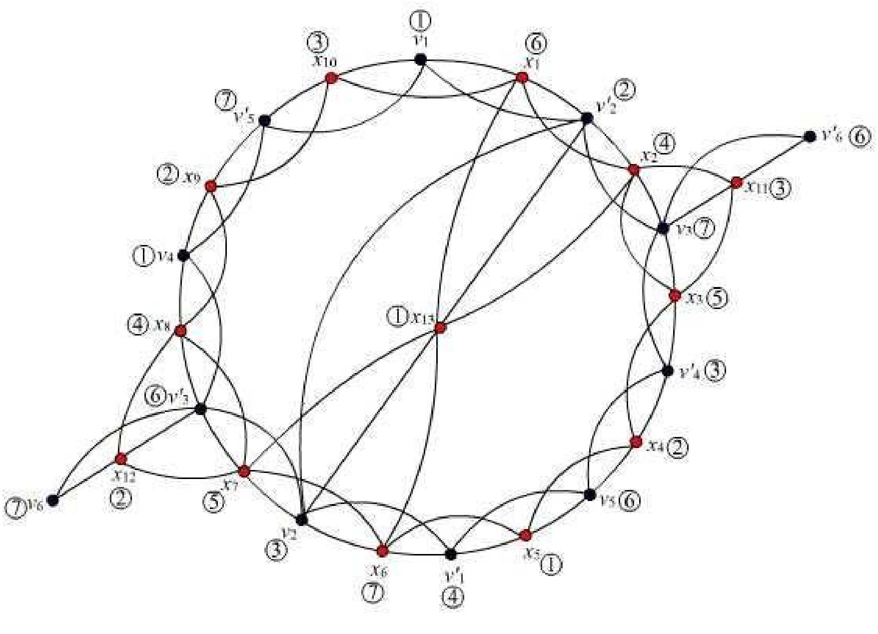

Figure 4 $\boldsymbol{T}[\boldsymbol{EDG}(\boldsymbol{T}_{5,1})]$

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The b - vertices x_{2n+3} , v'_2, v_2 x_2, x_{n+3}, v'_3 v_3 due of the color classes ($c_1, c_2, c_3, c_4,$ c_5 , c_6 and c_7) respectively produces the b - chromatic coloring when n ≡ 0(mod 2) and the b - vertices x_{2n+3} , v'_2, v_2 x_2, x_{n+2}, v'_3 v_3 due of the color classes $c_1, c_2, c_3, c_4,$ c_5 , c_6 and c_7 respectively produces the b - chromatic coloring when n ≡ 1(mod 2).

Hence $\varphi(T[EDG(T_{n,1})]) = 7 , n \geq 4$

5. b-chromatic number of total graph of extended duplicate graph of sunlet graph

An n-sunlet graph by adding n pendent edges to a cycle graph S_n, a graph on 2n vertices is created. EDG(S_n) represents the extended duplication graph of a sunlet graph, which has 2n vertices and 2n+1 edges.

Theorem 5.1

The b-chromatic number of total graph of extended duplicate graph of sunlet graph is $\varphi(T[EDG(S_n)]) = 7 , n \geq 4$.

Proof:

Let the extended duplicate graph of Sunlet's entire graph T[EDG(S_n)] represents S_n, which has 8n+1 vertices and 18n+9 edges with a maximum degree of 8 and a minimum of 2. It has 2n degree 2 vertices, 2(n-1) degree 4 vertices, 2 degree 5 vertices, 4n-6 degree 6 vertices, 4 degree 7 vertices, and 3 degree 8 vertices.

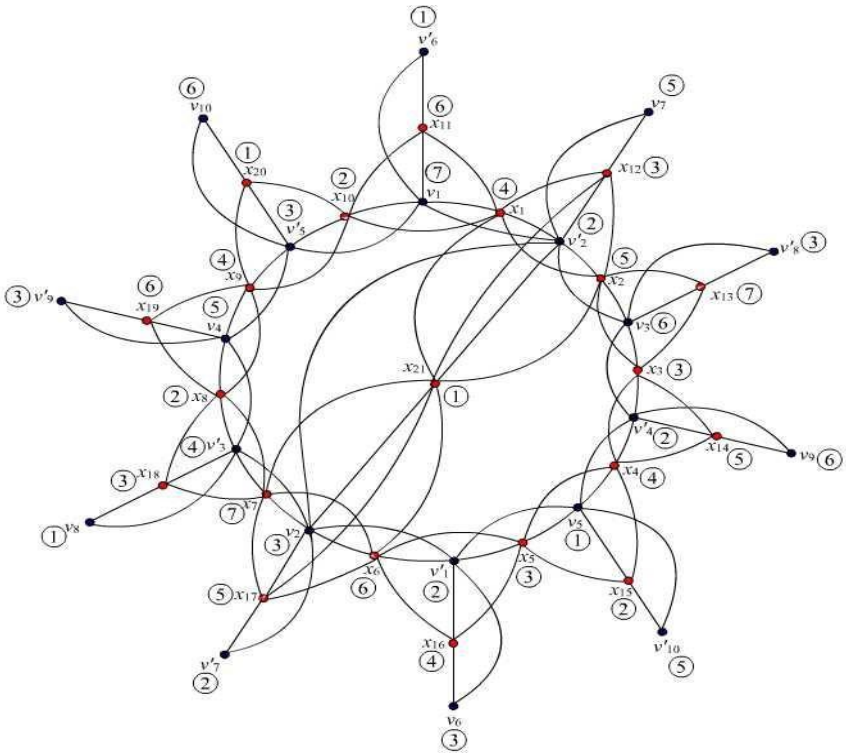

Figure 5 $\boldsymbol{T[EDG(S_5)]}$

As $\Delta(T[EDG(S_n)]) = 8$ by (1.1), $\varphi(T[EDG(S_n)]) \leq 9$. If $\varphi(T[EDG(S_n)]) \leq 9$ then $T[EDG(S_n)]$ must have nine vertices of degree 8, which is not possible, as $T[EDG(S_n)]$ has only three vertices of degree 8. Consequently, $\varphi(T[EDG(S_n)]) \neq 9$. If $\varphi(T[EDG(S_n)]) = 8$ then $T[EDG(S_n)]$ must have eight vertices of degree 7, which is also not possible. Consequently, $\varphi(T[EDG(S_n)]) \neq 8$.

Due to adjacency of vertices in $T[EDG(S_n)]$ at most seven b- vertices can be generated for any proper coloring. Consider the following 7-coloring ($c_1, c_2, c_3, c_4, c_5, c_6, c_7$) of $T[EDG(S_n)]$, assign the color c_1 to x_{4n+1}, assign the color c_2 to $v_2^{'}$, assign the color c_3 to v_2, assign the color c_4 to x_1, assign the color c_5 to x_2 , assign the color c_6 to x_{n+1}, assign the color c_7 to x_{n+2} and for the remaining vertices assign one of the allowed color - such color exists. The b-vertices x_{4n+1}, $v_2^{'}$, v_2, x_1, x_2,v_2, x_{n+1}, x_{n+2} for the color classes c_1, c_2, c_3, c_4, c_5, c_6, c_7 respectively produces the b - chromatic coloring.

Hence $\varphi(M[EDG(S_n)]) = 7 , n \geq 4$.

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