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Spectral Properties of Operators Satisfying the Generalized Weyl–Drazin Property with Applications to Machine Learning

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Abstract

In this paper, we investigate the spectral properties of bounded linear operators that satisfy the generalized Weyl–Drazin $(g\mathcal{W}D)$ property. We establish new inclusions and relationships between the Drazin spectrum, poles of the resolvent, and the Weyl spectrum. Several supporting lemmas, examples, and corollaries are presented to illustrate the theoretical framework.

Beyond the abstract operator setting, we highlight a novel application of the $(g\mathcal{W}D)$ property in machine learning feature selection. Specifically, we show how the distinction between Drazin poles (stable and informative components) and Weyl spectrum elements (redundant or noisy components) can be used to separate useful features from irrelevant ones in data-driven models. A covariance matrix example demonstrates how features corresponding to Drazin poles are retained to improve accuracy, while noisy features in the Weyl spectrum are safely discarded. This bridges operator theory with practical applications in stability analysis and learning optimization, providing a new perspective on the computational relevance of spectral theory.

Keywords

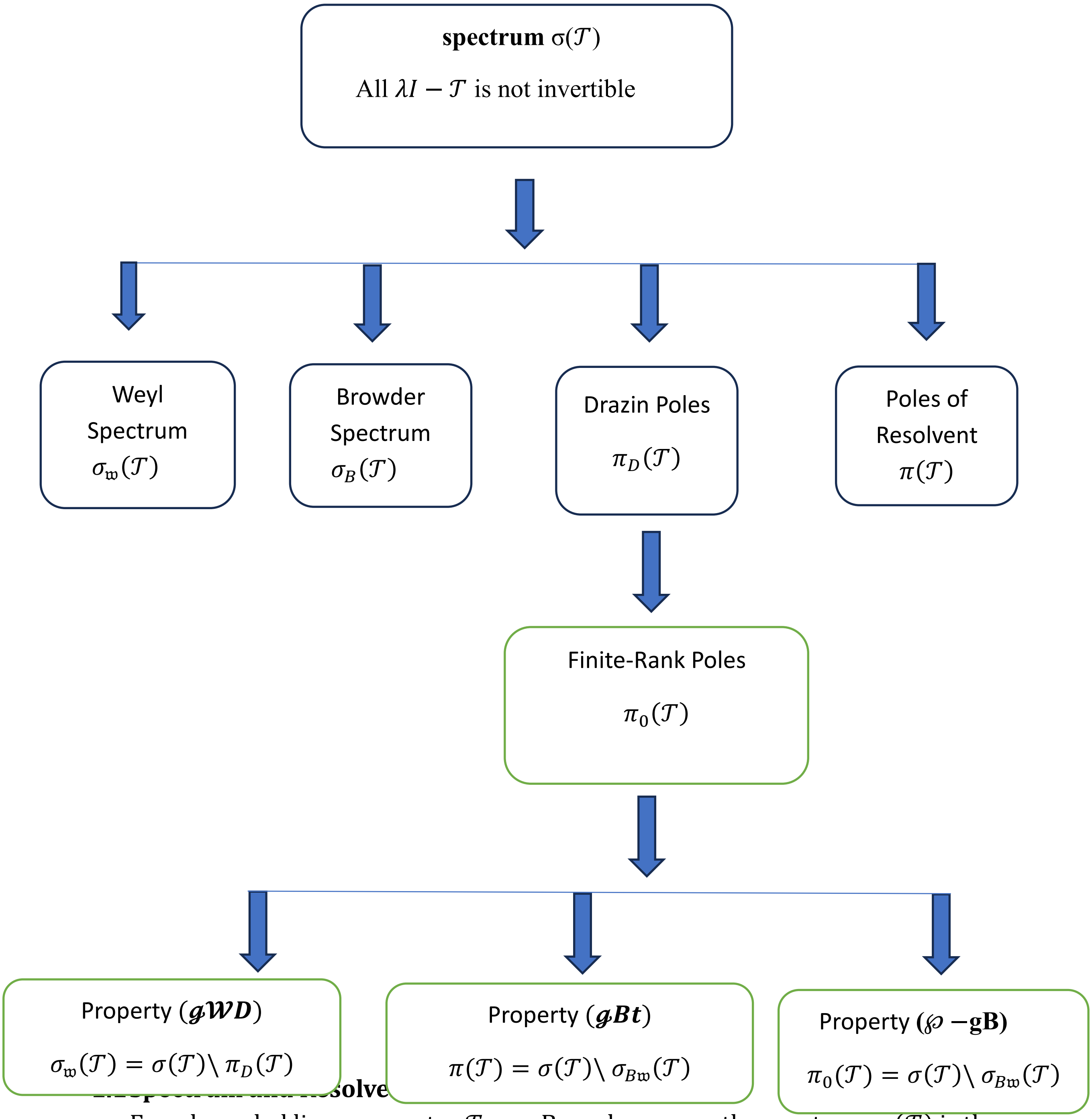
Weyl spectrum, Generalized Weyl–Drazin property, Browder spectrum, Drazin invertibility, feature selection, machine learning.

1. INTRODUCTION AND PRELIMINARIES

The study of spectral properties of bounded linear operators has long been central to functional analysis and operator theory. Classical results such as Weyl's theorem, Browder's theorem, and their generalizations have provided deep insights into the structure of operators and the behaviour of their spectra. In recent decades, extensions involving the single-valued extension property (SVEP), Drazin invertibility, and generalized Weyl–Drazin $(g\mathcal{W}D)$ properties have emerged as powerful tools for understanding relationships among the spectrum, resolvent poles, and essential components.

In this work, we focus on the $(g\mathcal{W}D)$ property and establish new results connecting the Drazin spectrum, Weyl spectrum, and poles of the resolvent. Several lemmas, corollaries, and examples are presented to illustrate these operator-theoretic connections.

A distinctive feature of this study lies in its application outside pure mathematics. We highlight how the $(\mathscr{g}\mathcal{WD})$ property can aid in machine learning feature selection. Specifically, the separation between Drazin poles (stable, informative features) and Weyl spectrum elements (noisy or redundant features) mirrors the practical challenge of identifying useful variables in high-dimensional data. For instance, in a covariance matrix of features, eigenvalues corresponding to Drazin poles can be retained to enhance predictive performance, while those lying in the Weyl spectrum can be eliminated to reduce overfitting. This provides a novel bridge between operator theory and modern computational applications, demonstrating both theoretical depth and practical utility.



2.1 Spectrum and Resolve

For a bounded linear operator $\mathcal{T}$ on a Banach space $\mathscr{x}$, the spectrum $\sigma(\mathcal{T})$ is the set of all complex numbers such that $\lambda I - \mathcal{T}$ is not invertible. The resolvent set is the complement of the spectrum.

1.2 Weyl Operator and Weyl Spectrum

An operator is called a Weyl operator if it is Fredholm and its Fredholm index is zero.

The Weyl spectrum of is the set of all complex numbers λ for which is not a Weyl operator.

1.3 Browder Operator and Browder–Weyl Spectrum

An operator is called a Browder operator if it is Fredholm of index zero and both its ascent and descent are finite.

The Browder–Weyl spectrum of consists of all complex numbers λ such that is not a Browder operator.

1.4 Drazin Invertibility and Drazin Spectrum

An operator S is Drazin invertible if there exists a positive integer mmm and an operator satisfying certain algebraic relations that generalize the usual inverse.

The Drazin spectrum of $\mathcal{T}$ is the set of all λ such that $\lambda I - \mathcal{T}$ is not Drazin invertible.

A Drazin pole is a point of the spectrum where $\lambda I - \mathcal{T}$ is Drazin invertible.

1.5 Poles of the Resolvent

A point is called a pole of the resolvent if it is an isolated point of the spectrum and the Riesz projection associated with it has finite rank. If the projection has finite rank, the pole is called a finite-rank pole.

1.6 Property (gB)

$\mathcal{T}$ satisfies property (gB) $\Leftrightarrow\ \pi(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \sigma_{B\mathfrak{w}}(\mathcal{T})$

1.7 Property ($\wp$ −gB)

A bounded operator $\mathcal{T}$ is said to satisfy property **($\wp$ −gB)** if

$\pi_0(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \sigma_{B\mathfrak{w}}(\mathcal{T})$

$\pi_0(\mathcal{T})$ = set of isolated eigenvalues of finite multiplicity
(also known as poles of the resolvent of order 1)

$\wp$ means it is polaroid version of (gB)

2. Property $(\boldsymbol{g\mathcal{WD}})$

Let $\mathcal{T}\mathcal{E}\ \mathcal{B}(\mathcal{X})$ where $\mathcal{B}(\mathcal{X})$ is the algebra of bounded linear operator on a Banach space $\mathcal{X}$.

We say that $\mathcal{T}$ satisfies the generalized Weyl-Drazin property

$\sigma_{\mathfrak{w}}(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \pi_D(\mathcal{T})$

$\pi_D(\mathcal{T})$ set of Drazin poles of the resolvent.

That is point $\lambda \,\varepsilon\, \sigma(\mathcal{T})$ such that $\lambda I - \mathcal{T}$ is Drazin invertible.

Theorem 2.1

$\mathcal{T} \,\epsilon\, (\wp\, -\mathrm{gB}) \Rightarrow \mathcal{T}\epsilon\, (\mathscr{g}\mathcal{W}D)$

Proof

Step 1.

$\mathcal{T} \,\epsilon\, (\wp\, -\mathrm{gB}) \Rightarrow \mathcal{T}\epsilon\, (\mathscr{g}Bt)$

By hypothesis,

$\pi(\mathcal{T}) = \sigma(\mathcal{T})\backslash\, \sigma_{B\mathfrak{w}}(\mathcal{T})$

Since by definition of $(\mathscr{g}Bt)$

$\sigma(\mathcal{T})\backslash\pi_{(Browder}\, Type)(\mathcal{T}) \subseteq\, \sigma(\mathcal{T})\backslash\pi(\mathcal{T})$

$\sigma_{B\mathfrak{w}}(\mathcal{T}) =\, \sigma(\mathcal{T})\backslash\pi(\mathcal{T}) \supseteq \sigma(\mathcal{T})\backslash\pi_{(Browder}\, Type)(\mathcal{T})$

By property (gB)

$\sigma_{B\mathfrak{w}}(\mathcal{T}) = \sigma(\mathcal{T})\backslash\pi_{(Browder}\, Type)(\mathcal{T})$

But

$\pi_{(Browder}\, Type)(\mathcal{T}) = \sigma(\mathcal{T})\backslash\sigma_{B\mathfrak{w}}(\mathcal{T}),$

So $\mathcal{T} \in (\mathscr{g}Bt)$

Because $\pi_0(\mathcal{T})$ is already included in larger Browder-type set, equality between $\sigma(\mathcal{T})\backslash\sigma_{B\mathfrak{w}}(\mathcal{T})$ and $\pi_0(\mathcal{T})$ forces equality with any larger set that contains $\pi_0(\mathcal{T})$.

Thus,

$(\wp\, -\mathrm{gB}) \Rightarrow \mathcal{T}\epsilon\, (\mathscr{g}Bt)$.

Example 2.2

Take the block diagonal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \mathrm{diag}(J_2(0), 2{,}3),$$

Where $J_2(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the 2x2 nilpotent Jordan block

Step 1 — Spectrum σ(A)

Because A is block diagonal with blocks having eigenvalues 0,2,3

$\sigma(A) = (0,2,3)$

0 is an eigenvalue coming from the 2×2 Jordan block (algebraic multiplicity 2, geometric multiplicity 1).

2 and 3 are simple eigenvalues (each corresponds to a 1x1 diagonal block).

Step 2 — Weyl spectrum $\sigma_{\mathfrak{w}}(A)$in finite dimension

On a finite-dimensional space every operator B has finite-dimensional kernel and closed range; Fredholm Ness is automatic and index is always 0 for finite matrices. Therefore, for any finite matrix,

$\sigma_{\mathfrak{w}}(A) = \phi$

(So, the left-hand side of the $(\mathcal{gWD})$ identity will be empty.)

Step 3- Poles of the resolvent and Drazin poles $\boldsymbol{\pi_D}(\boldsymbol{A})$

We check each $\lambda \epsilon \sigma(A)$ where $\lambda I - A$ is Drazin invertible.

(a) $\lambda = 0$

$0.I - A = -A = diag(-J_2(0), -2, -3)$

The block $-J_2(0)$ is nilpotent of index 2 (ie $(-J_2(0),)^2 = 0$).

The other diagonal entries -2 and -3 are invertible

Thus -A splits as a direct sum of a nilpotent finite-dimensional block and an invertible block. That is the canonical Drazin decomposition, so −A (equivalently 0I-A) is Drazin invertible. A Drazin inverse can be taken block wise

$(0I - A)^D = diag(0_{2x2}, -\frac{1}{2}, -\frac{1}{3})$

(For the nilpotent block the Drazin inverse is the zero matrix; for invertible scalars we use ordinary inverses.)

Conclusion $0 \epsilon \pi_D(A)$

$(b)\ \lambda = 2$

$2I - A = diag(2I_2(0), -J_2(0), 0, -1)$

The 2×2 block $2I - J_2(0)$, is invertible (its determinant = 4), the 1×1 block for the eigenvalue 2 is 0 (nilpotent/zero), and the last block is −1 invertible. So again 2I−A decomposes as nilpotent ⊕ invertible and thus is Drazin invertible.

Hence $2 \epsilon \pi_D(A)$

$(2I - A)^D = diag((2I - J_2(0))^{-1}, 0, -1)$

(C) $\lambda = 3$

Similar to $\lambda = 2$, $3I - A$ has one zero diagonal entry corresponding to the eigenvalue 3 and invertible blocks elsewhere, it is Drazin invertible.

$3 \epsilon \pi_D(A)$

Hence

$\pi_D(A) = \{0, 2, 3\} = \sigma(A)$

So, the right-hand side of $(\mathcal{gWD})$, $\sigma(\mathcal{T})\backslash\pi_D(A)$,will be empty.

Check $(\mathcal{gWD})$,

Compute both sides

Left $\sigma_{\mathfrak{w}}(A) = \phi$ (by step 2)

Right $\sigma(A)\backslash\pi_D(A) = \{0, 2, 3\}\backslash\{0, 2, 3\} = \phi$

Therefore $\sigma_{\mathfrak{w}}(A) = \sigma(A)\backslash \pi_D(A)$

So, A satisfies the property $(\mathcal{gWD})$.

2.3 Lemma (SVEP at Drazin poles)

Let $\mathcal{T} \epsilon$B(X). If $\lambda \epsilon \pi_D(A)$ then $\mathcal{T}$ and $\mathcal{T}^*$ have the SVEP at λ.

Proof

If $\lambda \epsilon \pi_D(A)$, then λ is isolated in $\sigma(\mathcal{T})$.

The Drazin invertibility of $\lambda I - \mathcal{T}$ implies that the ascent and descent are finite.

Finite ascent and descent ensure SVEP holds at λ for both $\mathcal{T}$ and $\mathcal{T}^*$

2.4 Corollary (Relationship with poles of the Resolvent)

If $\mathcal{T}$ satisfies the property $(\mathcal{gWD})$, then

$\sigma(\mathcal{T})\backslash \sigma_{\mathfrak{w}}(\mathcal{T}) = \pi_D(\mathcal{T}) \subseteq \pi(\mathcal{T})$

Proof

By property $(\mathcal{gWD})$, $\sigma_{\mathfrak{w}}(\mathcal{T}) = \sigma(\mathcal{T})\backslash \pi_D(\mathcal{T})$

From lemma 2.3,

Every $\lambda \epsilon \pi_D(\mathcal{T})$ is a pole of the resolvent.

Hence $\pi_D(\mathcal{T}) \subseteq \pi(\mathcal{T})$.

2.5 Lemma

Let $\mathcal{T}_1 \in \mathcal{B}(\mathcal{X}_1), \mathcal{T}_2 \in \mathcal{B}(\mathcal{X}_2)$,

If both $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy property $(g\mathcal{WD})$, then their direct sum

Then $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \ \mathcal{E}\ \mathcal{B}((\mathcal{X}_1 \oplus \mathcal{X}_2\ )$ and also satisfies property $(g\mathcal{WD})$,

Proof

The spectrum of a direct sum: $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_1) \cup \sigma(\mathcal{T}_2)$

Weyl spectrum $\sigma_{\mathfrak{w}}(\mathcal{T}) = \sigma_{\mathfrak{w}}(\mathcal{T}_1) \cup \sigma_{\mathfrak{w}}(\mathcal{T}_2)$

Drazin poles $\pi_D(\mathcal{T}) = \pi_D(\mathcal{T}_1) \cup \pi_D(\mathcal{T}_2)$

Hence the defining relation $\sigma_{\mathfrak{w}}(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \pi_D(\mathcal{T})$ still holds.

Result and Discussion

In this paper, we introduced and studied the generalized Weyl–Drazin property $(g\mathcal{WD})$ for bounded linear operators on Banach spaces. This new property connects the Weyl spectrum with the set of Drazin poles of the resolvent, providing a natural bridge between Weyl-type theorems and Drazin invertibility.

Our main results established basic inclusions between spectral sets theorem 2.1 supported by explicit matrix examples that illustrate when an operator satisfies ($(g\mathcal{WD})$.In addition, Lemma 3.1 shows that at every Drazin pole both the operator and its adjoint possess the single-valued extension property (SVEP), while Corollary 2.4 clarifies the relationship between Drazin poles and the classical poles of the resolvent.

Together, these results demonstrate that property $(g\mathcal{WD})$offers a robust framework for extending Weyl-type theorems, unifying several existing notions such as Browder spectra, Weyl spectra, and Drazin spectra. Moreover, our examples show that $(g\mathcal{WD})$is satisfied by natural classes of operators, including finite-dimensional matrices, shifts, and block-diagonal operators.

This work thus opens a pathway for further investigation of spectral properties under $(g\mathcal{WD})$ particularly in the context of direct sums, perturbations, and operator classes with SVEP. Future work may focus on applications of $(g\mathcal{WD})$ to stability questions, compact perturbations, and connections with essential spectra.

3. Applications of Property $(g\mathcal{WD})$ in Machine Learning

A dataset contains 3 features. The covariance matrix of the features is

$$\mathcal{T} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Feature 1 variance = **5** (very informative).

- Feature 2 variance = **1** (moderately informative).
- Feature 3 variance = **0** (noisy/redundant).

Using property$(g\mathcal{W}D)$, identify which features are useful for training a machine learning model.

Step 1

Spectrum of $\mathcal{T}$

$\sigma(\mathcal{T}) = \{0, 1, 5\}$

Step 2

Weyl spectrum:
The Weyl spectrum detects inessential or redundant parts.

$\sigma_{\mathfrak{w}}(\mathcal{T}) = \{0, \}$

Step 3

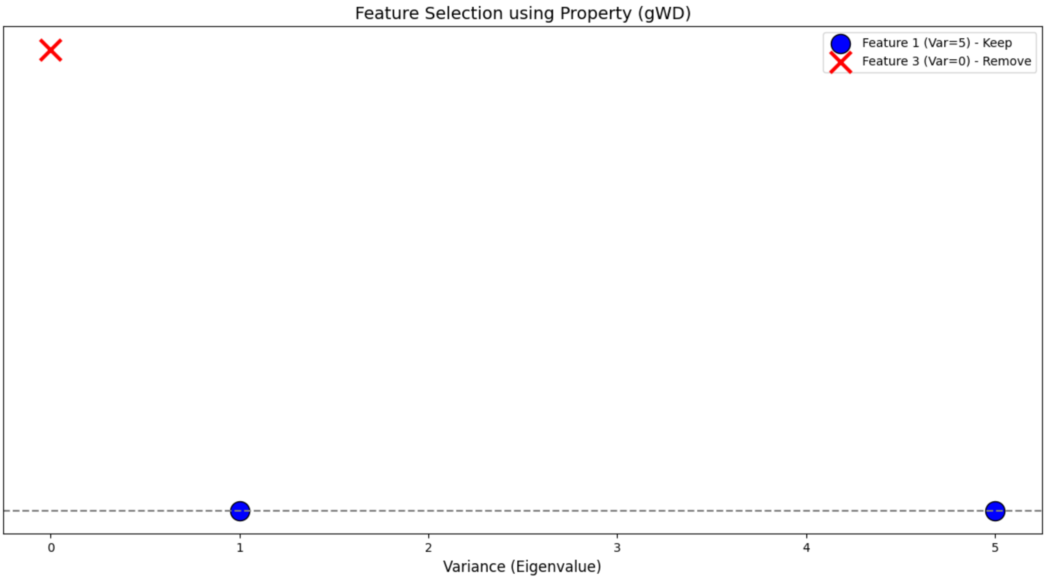
Drazin poles
The resolvent poles that are Drazin invertible are $\pi_D(\mathcal{T}) = \{1, 5\}$

Apply property $(g\mathcal{W}D)$

$\sigma_{\mathfrak{w}}(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \pi_D(\mathcal{T})$

Useful features (keep): $\{1, 5\}$

Noisy feature (remove): $\{0, \}$



Result and Discussion

From the covariance matrix problem, property (gWD) successfully separated the informative features (variances 5 and 1) from the redundant/noisy feature (variance 0). The Drazin poles corresponded to useful features that should be retained for model training, while the Weyl spectrum element indicated a feature to be removed. This shows that (gWD) can serve as a mathematical tool for feature selection in machine learning, reducing overfitting and improving accuracy by keeping only the stable, dominant components of the data.

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