UtilitasMathematica

ISSN 0315-3681 Volume 122 (2), 2025

Minimum Tree t-Spanners on Fuzzy Ratio Labelled General Graphs - NP Complete

R. Mathu Pritha [1][0000-0001-5419-0581] and A. Amutha [2] [0000-0002-7123-3026]

[1] Department of Mathematics, The American College Madurai-625 002, Tamil Nadu, India
[2] PG & Research Department of Mathematics, The American College Madurai-625 002, Tamil Nadu, India

vmarmp@gmail.com, amutha@americancollege.edu.in

Abstract. Tree t-spanners are trees that span a graph that involve distances and are useful in designing telecommunication, electrical networks, civil network planning, and more. Extending the idea of tree t–spanner in fuzzy graphs is the concept of this paper. The study focuses on tree t-spanners of ratio-labelled fuzzy graphs using the Breadth-First Search (BFS) algorithm, which is crucial for optimizing communication and routing efficiency in networks. Ratio Labelling (RL), a novel method for fuzzifying crisp graphs, which labels vertices and edges with the particular relation for σ and μ. The paper examines the stretch factor 't' of a spanning tree on ratio-labelled fuzzy graphs (RLFG). The paper extends to find tree t-spanners of arbitrary ratio-labelled fuzzy graphs. The study reveals that determining tree spanners of an arbitrary ratio-labelled fuzzy graph is NP-complete. Also analyzed the relationship between group of friends on social media using RL. The minimum spanner t of this ratio labelled fuzzy graph helps to identify the people who are less communicative and the possible two friends who can communicate in future are identified using link prediction method.

Keywords: Fuzzy graph, ratio labelling, tree t-spanner, complete graph, complete bipartite graph, general graph, NP-complete.

1 Introduction

Fuzzy graph theory originated with Kaufmann's 1973 work, which was based on Zadeh's work on fuzzy relations [17, 30]. However, Rosenfeld's 1975 publication is considered as the cornerstone of the field. Rosenfeld significantly expanded the theory by defining fuzzy relations in fuzzy sets and exploring fundamental concepts of graphs such as paths, cycles, and connectedness within the fuzzy framework [27]. Later, Nagoor Gani and Rajalakshmi introduced the concept of labelling in fuzzy graphs. Labelling of fuzzy graph introduced by A. Nagoor Gani and D.Rajalakshmi [25]. The work by Mathew Varkey, T. K., and Sreena T. D. on evidence labelling of fuzzy graphs examines the fuzziness of crisp graphs [21, 22]. The spectrum of Laplacian matrix

---

RL-Ratio Labelling
RLFG - Ratio Labelled Fuzzy Graph

for complete fuzzy graphs was analysed by Amutha A et al. [3]. The concept of ratio labelling was introduced by A. Amutha, and R. Mathu Pritha to verify the admittance of fuzziness in crisp graphs [2]. Further analyzed interconnection networks graphs for admittance of fuzziness by A. Amutha, R. Mathu Pritha [4]. Fuzzy graphs have their utility in medical field, telecom- munication, traffic light control. Particularly application of fuzzy graph in traffic congestion is done by R. Myna and by P.Sinthamani [24, 29].

A graph spanner of a graph G is a subgraph that approximates the shortest path distances in G within a certain distortion, which can be either multiplicative or additive. Spanners were introduced by Peleg and Schaffer [11, 12]. The effectiveness of a network is often evaluated using the stretch factor. Minimum-weight spanners are particularly useful in network design and communication networks. Different types of spanners can be created depending on the specific goals of choosing the subnetwork [1, 5, 18, 26]. Ioannis Papoutsakis examined the tree spanners of bounded degree graphs [16]. Tree spanners in Chordal graphs were discussed by Brandstadt and et al. [7]. A genetic algorithm with interval type 2 fuzzy arc length to find a fuzzy mini- mum spanning tree was found by Dey, A., Pal, A. et al., in 2020 [13]. The minimal spanning tree problem was compared with different algorithms by Archana A. Deshpande and Onkar K. Chandhari in 2020 [14]. Katick Mohanta, Arindam Dey, Narayan C. Debath, and Anita Pal in 2019 formulated an algorithm for a minimum spanning tree in an intuitionistic fuzzy graph [23]. Tree 3-spanners on generalized prisms of graphs were analyzed in 2022 by Renzo Gomez, Flavio K. Miyazawa, and Yoshiko Wakabayshi [15]. Fernanda Couto et al. in 2022 discussed the strategies for generating tree spanners [9]. The characteristics of tree t-spanners of graphs with a few P4's were studied by Fernando Couto et al. in 2022[10]. Local routing in a tree metric 1-Spanner was studied by Brankovic M, Gudmundsson J, and Renssen AV in 2022[8]. In 2023, Francis Remigius Perpetua Mary et al. discussed the minimum spanning tree problem under the Fermatean fuzzy environment [20].

Fuzzy graphs, where edges have degrees of membership between 0 and 1, are gaining attention because of their ability to model uncertain relationships. This 'fuzziness' is often captured through vertex and edge labeling. A fuzzy graph, G = (V, E), uses fuzzy sets to define edges, reflecting the strength of the connections. Blue et al. have proposed a classification system for these graphs based on various properties as follows [6].

"Type: (I) crisp vertices with fuzzy edges, Type (II) crisp vertices and edges with fuzzy connectivity, Type (III) fuzzy vertices with crisp edges, Type (IV) crisp graphs with fuzzy weights, representing uncertainty in vertex and edge properties, and Type (V) fuzzy sets of crisp graphs, or fuzzy compositions. "are the classification system of graphs according to M. Blue, B. Bush, J. Puckett.

Type IV is particularly useful for modeling scenarios where connections are well defined, but their attributes are uncertain. Using Type IV classification, we introduced Ratio Labelling (RL) to analyze the fuzziness of crisp graphs. This method uses graph parameters to assign membership values (between 0 and 1) to vertices and edges, effectively capturing the graph structure and allowing us to determine the degree of fuzziness existing in the graph. A similar approach was used earlier by Mathew Varkey T. K. and Sreena T. D. on evidence labelling[21, 22]. They gave a particular definition for σ and μ and verified the admissibility of fuzziness. According to evidence labelling, all graphs are fuzzy graphs. In this paper, we generalize the concept of fuzzification in general crisp graphs under RL. But under RL all graphs are not fuzzy graphs. Admissibility of fuzziness by some of the interconnection networks were analyzed using RL [4]. The graph that admits fuzziness under RL ensures a strong connectivity between the vertices. The novelties and effectiveness of ratio labeling are listed as follows.

- In RL, labels are assigned in the form of ratios that express the relative value between two parameters. Such ratios serve to indicate the comparative strength, significance, or impact of a vertex or an edge within a graph.

- RL can dynamically adjust based on the relative importance of a vertex or edge compared to others in the graph, allowing for more context-sensitive labelling.
- RL help in highlighting fine variations in relationships or characteristics.
- The effectiveness of ratio labelling depends on precise and reliable data for generating meaningful ratios. When the data is limited, missing, or contains noise, the resulting ratios may lose their accuracy and dependability.

Since trees provide unique paths between nodes, they simplify message routing in distributed systems, making them highly useful in areas such as computer networks, sensor networks, ect. A tree spanner serves as an approximation of the original graph's distances, typically within a bounded factor. This feature is advantageous in scenarios where exact distances are not required, but efficient approximations are sufficient. The study of tree spanners in fuzzy graphs is a growing field in graph theory that adapts the well-established concept of tree spanners from crisp graphs. This has inspired us to further explore tree spanners within the framework of ratio-labelled fuzzy graphs. The paper focuses on constructing minimum tree spanners of the ratio labelled fuzzy graphs like cycle, path, complete graph and complete bipartite graph in section 3. The tree spanners of ratio labelled general crisp graphs are discussed in section 4.

2 Basic Concepts

A fuzzy subset of a non-empty set S is a mapping $\sigma : S \to [0, 1]$ which assigns to each element x in S a degree of membership $0 \le \sigma(x) \le 1$. A fuzzy graph G : (σ, μ) is a pair of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$, where for all x, y ∈V,

$$\mu(x, y) \le \sigma(x) \wedge \sigma(y) .$$

where $\wedge$ stands for minimum. A fuzzy relation μ on S is said to be symmetric if

$$\mu(x, y) = \mu(y, x) \text{ for all x, y} \in \text{S and}$$

$$\sigma^* = supp(\sigma) = \{u \epsilon S: \sigma(u) > 0\}. \mu^* = supp(\mu) = \{(u, v) \epsilon S \times S: \mu(u, v) > 0)$$

In $G: (\sigma, \mu)$, the order of G is

$$p = \sum_{x \in S} \sigma(x) .$$

If $\mu(x, y) > 0$ then x and y are called neighbours, x and y are said to lie on the same edge e. The neighbourhood of a vertex $v \in S$ is a set of all vertices which are neighbours of v denoted by $N(v)$. A fuzzy graph (σ', μ') is a fuzzy subgraph or a partial fuzzy subgraph of (σ, μ) if $\sigma' \subseteq \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) \le \sigma(u)$ for every $u \in S$ and $\mu'(e) \le \mu(e)$ for every $e \in E$.

A fuzzy graph (σ', μ') is a fuzzy spanning subgraph of (σ, μ) if $\sigma' = \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) = \sigma(u)$ for every $u \in S$ and $\mu'(e) \le \mu(e)$ for every $e \in E$. Let G: (σ, μ) be a fuzzy graph. The degree of a vertex 'u' is defined as $d(u) = \sum \mu(u, v)$, $u \neq v$, $v \in S$. It is also denoted as $d_G(u)$.

A fuzzy graph G is said to be a strong fuzzy graph if

$$\mu(x,y) = \sigma(x) \wedge \sigma(y) \text{ for all } (x,y) \text{ in } \mu^*$$

A fuzzy graph G is said to be a complete fuzzy graph if

$$\mu(x,y) = \sigma(x) \wedge \sigma(y) \text{ for all } x,y \text{ in } \sigma^*$$

A fuzzy graph is said to be regular if every vertex is of the same degree. In a fuzzy graph $G:(\sigma,\mu)$, a path is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0,\ 1 \le i \le n$. Here, '$n$' is called the length of the path. The consecutive pairs (v_{i-1}, v_i) are called arcs of the path. If u, v are nodes in G and if they are connected using a path then the strength of that path is defined as $\wedge_{i=1}^{n} \mu(v_{i-1}, v_i)$. If u and v are connected using paths of length 'k' then $\mu^k(u,v)$ is defined as

$$\mu^k(u,v) = \sup\{\mu(u,v_1) \wedge \mu(v_1,v_2) \wedge \ldots \wedge \mu(v_{k-1},v) : u, v_1, \ldots, v_{k-1}, v \in S\}$$

A node u, is said to be an isolated node if $\mu(u,v) = 0$ for all $u \neq v$.

The fuzzy distance between two nodes u and v is defined as

$$d_f(u,v) = \wedge \sum \{\wedge(\sigma(u), \sigma(v)) \times \mu(u,v)\}.$$

A spanning subgraph G' of G is a t-spanner for G if, for every pair of nodes v and w,

$$d_{G'}(v,w) \le t \times d_G(v,w).$$

3 Main Results

Tree t-spanners are spanning trees of a graph that involve shortest distances and are useful in designing telecommunication, electrical networks, civil network planning, and more. It is necessary to find an efficient network that connects all the vertices of the communication network. The tree t-spanner preserves the length of the shortest path in a graph. The paper focuses on finding the minimum tree t-spanner of RLFGs. The vertices and edges of crisp graphs are labelled with particular definitions σ and μ respectively, known as Ratio Labelling (RL), enabling analysis of fuzziness. The study investigates tree t-spanners for RLFG using the BFS algorithm and determines bounds for the stretch factor t. Limited existing results on tree t-spanners for fuzzy graphs were found in the literature review. The extension of tree t-spanners to ratio- labelled arbitrary fuzzy graphs is a significant milestone in graph theory. The existence of a tree t-spanner in ratio-labelled arbitrary fuzzy graphs was investigated. As, BFS algorithm guarantees minimum stretch, the spanning tree for RLFG was found using BFS algorithm without considering the weightages of vertices and edges. The stretch will be minimum if we start the BFS algorithm with a vertex having maximum degree as root vertex.

Definition 3.1. Let $G(V,E)$ be a fuzzy weighted graph. T (V, E') is a fuzzy tree spanner of G, $E' \subset E$, if for each pair, $u, v \in V$ there is a path between u and v in T with length $d_f^T(u,v)$ atmost t times the shortest path distance $d_f(u,v)$ between u and v in G.

$$i.e.\quad d_f^T(u,v) \le t \times d_f^G(u,v)$$

Definition 3.2. A minimum spanner 't' of a fuzzy graph G is defined as

$$t(G) = \min\{t(T) : T \text{ is a spanning tree of } G\}.$$

Definition 3.3. A spanning tree T is called a minimum tree spanner if

$$t(T) \le t(T') \text{ for all spanning } T' \text{ of } G.$$

Example 3.4.

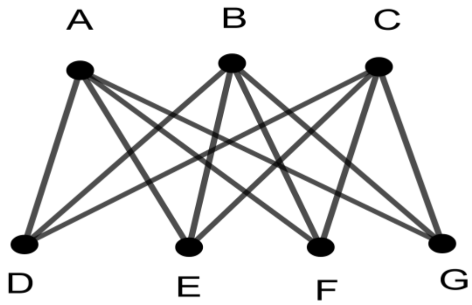


Figure 1: Bipartite graph $K_{3,4}$

G is a complete bipartite graph , $K_{3,4}$ with $V_1 = \{A, B, C\}$; $V_2 = \{D, E, F, G\}$
The vertices are labelled as

$$\sigma(\text{A}) = \frac{|N(\text{A})|}{|E|} = \frac{4}{12} = \frac{1}{3} = \sigma(\text{B}) = \sigma(\text{C});$$

$$\sigma(D) = \sigma(E) = \sigma(F) = \sigma(G) = \frac{3}{12} = \frac{1}{4}$$

and $\mu: E \rightarrow [0, 1]$ defined as

$$\mu(u, v) = \frac{\max[\sigma(u), \sigma(v)]}{\sum_{u \in V} \sigma(u)} = \frac{\max\left\{\frac{1}{3}, \frac{1}{4}\right\}}{2} = \frac{1}{6}$$

Here, $\mu(u, \text{v}) = \frac{1}{6} < \frac{1}{4} = \sigma(u) \wedge \sigma(v)$ for all u, v. Hence, $K_{3,4}$ is a fuzzy graph under RL. Let us find the tree spanner for $K_{3,4}$.

Now, $d_f^G(u, v) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$, $for\ u \in V_1, v \in V_2\ \ and$

$$d_f^G(u, v) = \frac{1}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{6} = \frac{2}{24} = \frac{1}{12}\ , for\ u, v \in V_1\ \ or\ \ V_2$$

By Breadth First Search algorithm with A as root vertex, the spanning tree T of $K_{3,4}$ is 4-ary tree.

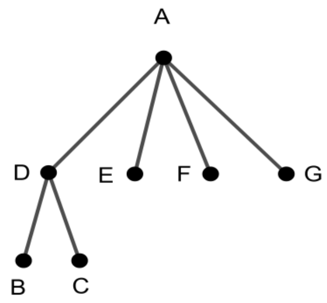


Figure 2: Spanning tree of $K_{3,4}$

The maximum stretch is between B to E, F, G and C to E, F, G. In T,

$$d_f^T(B,E) = d_f^T(B,D) + d_f^T(D,A) + d_f^T(A,E) = \frac{1}{4}\times\frac{1}{6}+\frac{1}{4}\times\frac{1}{6}+\frac{1}{4}\times\frac{1}{6} = \frac{3}{24} = \frac{1}{8} = d_f^T(B,F) = d_f^T(B,G);$$

Similarly, $d_f^T(C,E) = \frac{3}{24} = \frac{1}{8} = d_f^T(C,F) = d_f^T(C,G);$

In RLFG $K_{3,4}$, $d_f^G(B,E) = \frac{1}{4}\times\frac{1}{6} = \frac{1}{24} = d_f^G(B,F) = d_f^G(B,G)$

$$d_f^G(C,E) = \frac{1}{4}\times\frac{1}{6} = \frac{1}{24} = d_f^G(C,F) = d_f^G(C,G)$$

Hence, $d_f^T(u,v) \le 3 \times d_f^G(u,v)\ for\ all\ u,v \in K_{3,4}$.

$K_{3,4}$ admits fuzzy tree 3- spanner. The stretch will be same for all other root vertices from V_1 and V_2.

Hence, the minimum spanner of RLFG $K_{3,4}$ is 3.

Theorem 3.5. *Let G be a RLFG* C_n, $n \ge 3$. The minimum tree t-spanner of G is (n − 1).

Proof: In C_n, $|V| = n$, $|E| = n$, and $|N(v)| = 2$, for all $v \in V$.

Now, labelling the vertices and edges of C_n using RL,

$$\sigma(v) = \frac{|N(v)|}{|E|} = \frac{2}{n} \quad \text{for all } v \in V, \quad \sum_{v\in V}\sigma(v) = \sum_{v\in V}\frac{2}{n} = 2 \quad \text{and}$$

$$\mu(u,v) = \frac{\max[\sigma(u),\sigma(v)]}{\sum_{v\in V}\sigma(v)} = \frac{1}{n}, \text{ for all } (u,v) \in E.$$

Distance between any two vertices in a cycle C_n varies from i= 1 to $\left\lfloor\frac{n}{2}\right\rfloor$.

Fuzzy distance between two vertices is

$$d_f(u,v) = \wedge\sum_1^i\{\wedge(\sigma(u),\sigma(v)) \times \mu(u,v)\}$$

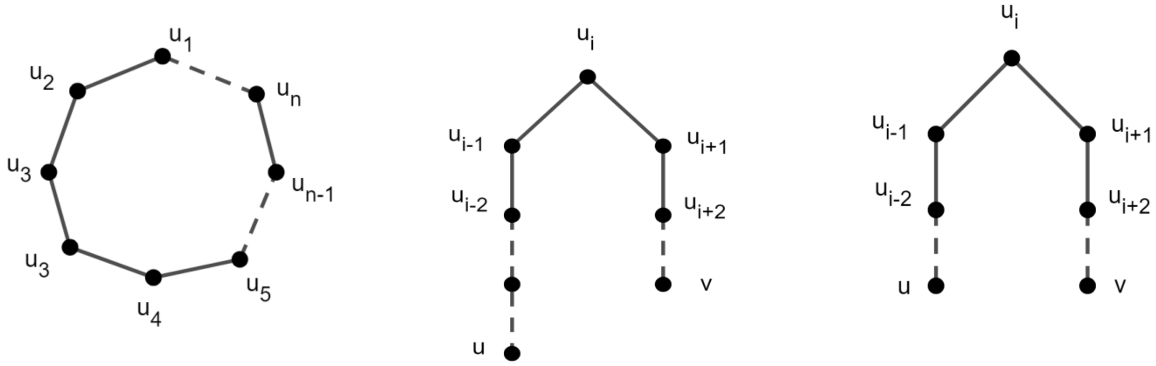
When u and v are at a distance i in a cycle C_n,

$$d_f(u,v) = \wedge\sum_1^i\left[\frac{2}{n}\times\frac{1}{n}\right] for\ all\ u,v \in V, i = 1\ to\ \left\lfloor\frac{n}{2}\right\rfloor$$

$$= \begin{cases} \frac{2}{n^2} & for\ i = 1 \\ 2\times\frac{2}{n^2} & for\ i = 2 \\ \dots & \\ \left\lfloor\frac{n}{2}\right\rfloor\times\frac{2}{n^2} & for\ i = \left\lfloor\frac{n}{2}\right\rfloor \end{cases}$$

Let the vertices of G be $u_1, u_2, \dots, u_n$. Let us find a spanning tree, T of G using BFS algorithm with root vertex, u_i, $1 \le i \le n$. Let, u, v be the two vertices which are at extreme distance in the spanning tree, T. That is, u, v are at maximum stretch. This u and v are adjacent in G. The spanning tree with root vertex u_i for n = even and odd is given in Figure 3. The distance between u and v in T is a path of length $n - 1$. As the graph is regular, σ, μ are same for all $u \in V$ and $(u,v) \in E$, respectively. The fuzzy distance between every pair of vertices which are at maximum stretch, is,

$$d_f^T(u,v) = (n-1) \times \frac{2}{n} \times \frac{1}{n} = (n-1) \times \frac{2}{n^2}$$



(a) Graph G (b) Spanning tree T, for n is even (c) Spanning tree T, for n is odd

Figure 3

Since $d_f^T(u,v)$ is equal for all $u_{i,}$, $1 \le i \le n$, the minimum of this maximum is

$$d_f^T(u,v) = (n-1) \times \frac{2}{n^2}$$

In G, u, v are the adjacent vertices. Hence the fuzzy distance of u, v in G is

$$d_f^G(u,v) = \frac{2}{n^2}$$

Now, $d_f^T(u,v) = (n-1) \times \frac{2}{n^2} \le (n-1) \times d_f^G(u,v)$

In general, for any two vertices u, v ,

$$d_f^T(u,v) \le (n-1) \times d_f^G(u,v)$$

Hence, RLFG C_n admits $(n-1)$- tree spanners.

As the graph is symmetric, the stretch is same for all u_i. Hence, the minimum tree t- spanner of RLFG (C_n) is $n-1$.

Theorem 3.6. Let G be a RLFG of the path P_n. Then minimum tree t-spanner of G is 1.

Proof: In P_n , $|V| = n$,$|E| = n-1$, and $|N(v)| = \begin{cases} 1, for\ pendant\ vertices \\ 2, for\ internal\ vertices \end{cases}$.

Now,$\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{n-1}, for\ pendant\ vertices \\ \frac{2}{n-1}, for\ internal\ vertices \end{cases}$.

$\mu(u,v) = \frac{\max[\sigma(u),\sigma(v)]}{\sum_{v \in V} \sigma(v)} = \frac{1}{n-1}$, for all $(u,v) \in E$,

The minimum tree t- spanner is obtained using BFS algorithm, and hence the minimum tree spanner of G is P_n itself. Hence,

$$d_f^T(u,v) = d_f^G(u,v)$$

RLFG(P_n) admits 1-fuzzy tree spanners.

Theorem 3.7. Let G be a RLFG K_n . The minimum tree t-spanner of G is 2.

Proof: In K_n , $|V| = n$, $|E| = \frac{n(n-1)}{2}$, and $|N(v)| = n-1$, for all $v \in V$.

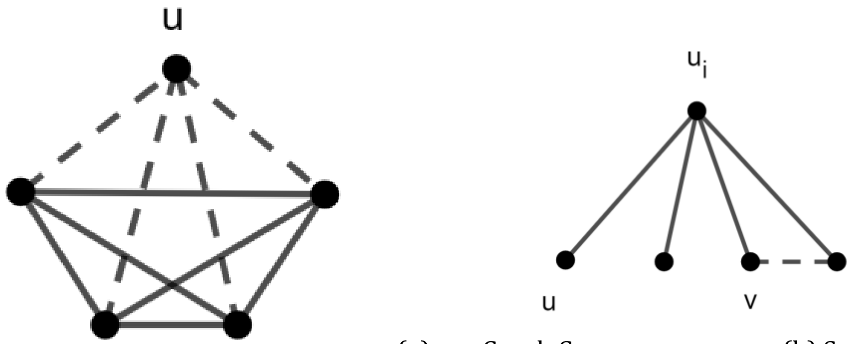
Hence,$\sigma(v) = \frac{|N(v)|}{|E|} = \frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}$, for all v.

$\mu(u,v) = \frac{\max[\sigma(u),\sigma(v)]}{\sum_{v \in V} \sigma(v)} = \frac{1}{n}$, for all $(u,v) \in E$

Fuzzy distance between any two vertices is,

$$d_f(u,v) = \wedge \sum \{\wedge(\sigma(u), \sigma(v)) \times \mu(u,v)\}$$

$$d_f^G = \wedge\left[\frac{2}{n}\times\frac{1}{n}\right] for\ all\ u, v$$
$$= \frac{2}{n^2}\ for\ all\ u, v.$$



(a) Graph G (b) Spanning tree

T , $K_{1,n}$

Figure 4

Let the vertices of G be $u_1, u_2, \dots, u_n$. Let us find a spanning tree, T of G using BFS algorithm with root vertex, $u_{i,}$, $1 \le i \le n$. The spanning tree T of G is $K_{1,n}$ as every vertex is adjacent to every other vertex in K_n. See Figure 4. Let u, v be the two vertices which are at extreme distance in the spanning tree, T. That is, u, v are at maximum stretch. Also, u and v are adjacent in G. The distance between u and v in T is a path of length 2. As the graph is regular, σ, μ are same for all $u \in V$ and $(u, v) \in E$, respectively. The fuzzy distance between every pair of vertices which are at maximum stretch is ,

$$d_f^T(u, v) = 2 \times \frac{2}{n^2} = \frac{4}{n^2}$$

Since $d_f^T(u, v)$ is equal for all $u_{i,}$, $1 \le i \le n$, the minimum of this maximum is

$$d_f^T(u, v) = \frac{4}{n^2}$$

In G, u, v are the adjacent vertices. Hence the fuzzy distance of u, v in G is

$$d_f^G(u, v) = \frac{2}{n^2}$$

Now, $d_f^T(u, v) = 2 \times \frac{2}{n^2} \le\ 2 \times d_f^G(u, v)$

In general, for any two vertices u, v ,

$$d_f^T(u, v) \le 2 \times d_f^G(u, v)$$ ■

Theorem 3.8. Let G be a RLFG $K_{m,n}, m \ne n$. Then minimum tree t-spanner of G is 3.

Proof: In $K_{m,n}$, $|V| = m + n$, where $|V_1| = m, |V_2| = n$, $|E| = m \times n$, and $|N(v)| = \begin{cases} n\ if\ v \in V_1 \\ m\ if\ v \in V_2 \end{cases}$.

Now, $\sigma(v) = \frac{|N(v)|}{|E|} = \begin{cases} \frac{1}{m}\ if\ v \in V_1 \\ \frac{1}{n}\ if\ v \in V_2 \end{cases}$.

Case(i)

For $m < n$, $K_{m,n}$ is a fuzzy graph under ratio labelling when $n \le 2m$

$$\mu(u, v) = \frac{\max[\sigma(u),\sigma(v)]}{\sum_{v\in V}\sigma(v)} = \frac{1}{2m}, \text{for all } (u, v) \in E,$$

Fuzzy distance between two vertices u, v is ,

$$d_f(u, v) = \wedge\sum\{\wedge(\sigma(u), \sigma(v)) \times \mu(u, v)\}$$

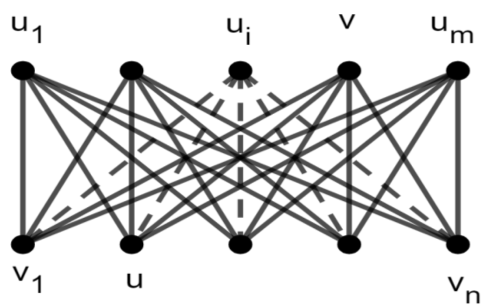
Let u and v be at a distance i, in $K_{m,n}$. The distance between any two vertices in $K_{m,n}$ is either 1 or 2. Hence i=1,2.

$$d_f^G(u, v) = \wedge \sum_1^i \left[\frac{1}{n}\times\frac{1}{2m}\right] for\ all\ u, v \in G \text{ and } i = 1, 2$$

$$= \begin{cases} \dfrac{1}{2mn} & for\ i = 1 \\ 2 \times \dfrac{1}{2mn} & for\ i = 2 \end{cases}$$

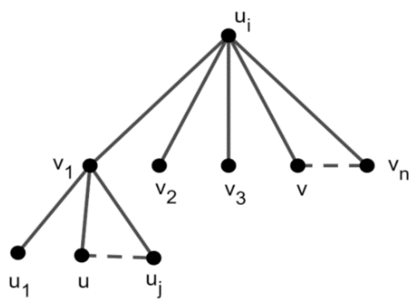
$$= \begin{cases} \frac{1}{2mn} & for\ i = 1 \\ \frac{1}{mn} & for\ i = 2 \end{cases}$$

Let the vertices of G be $u_1, u_2, \dots, u_m \in V_1$ and $v_1, v_2, \dots, v_n \in V_2$. Let us find a spanning tree, T of G using BFS algorithm with root vertex from either V_1 or V_2. But to get a minimum spanning tree the root vertex should be of maximum degree. As $m < n$, $u_1, u_2, \dots, u_m \in V_1$ are of maximum degree. Let us choose vertices from $u_1, u_2, \dots, u_m \in V_1$ as root vertex. Then T is a n-ary tree. See Figure 5. Let u, v be the two vertices which are at extreme distance in the spanning tree, T.



(a) Graph G

(b)



Spanning Tree T of G, $1 \le j \le m, i \ne j$

Figure 5

That is, u, v are at maximum stretch. The distance between u and v in T is a path of length 3. Also, u and v are adjacent in G. The fuzzy distance between every pair of vertices which are at maximum stretch is,

$$d_f^T(u, v) = 3 \times \frac{1}{2mn} = \frac{3}{2mn}$$

In G, u, v are adjacent vertices. Hence the fuzzy distance of u, v in G is

$$d_f^G(u, v) = \frac{1}{2mn}$$

$$\text{Now, } d_f^T(u, v) = \frac{3}{2mn} \le 3 \times d_f^G(u, v)$$

In general, for any two vertices u, v ,

$$d_f^T(u, v) \le 3 \times d_f^G(u, v) \text{ for } m < n$$

Case(ii)

For $m > n$, $K_{m,n}$ is a fuzzy graph under ratio labelling when $m \le 2n$

$$\mu(u, v) = \frac{\max[\sigma(u), \sigma(v)]}{\sum_{v \in V} \sigma(v)} = \frac{1}{2n}, \text{ for all } (u, v) \in E$$

Let u and v be at a distance i, in $K_{m,n}$. The distance between any two vertices in $K_{m,n}$ is either 1 or 2. Hence i=1,2.

Fuzzy distance between two vertices,

$$d_f(u, v) = \wedge \sum \{\wedge(\sigma(u),\ \sigma(v)) \times \mu(u, v)\}$$

$$d_f^G(u, v) = \wedge \sum_1^i \left[\frac{1}{m} \times \frac{1}{2n}\right] for\ all\ u, v \text{ and } i = 1, 2$$

$$= \begin{cases} \frac{1}{2mn} & for\ i = 1 \\ 2 \times \frac{1}{2mn} & for\ i = 2 \end{cases}$$

As $m > n$, $v_1, v_2, \dots, v_n \in V_2$ are of maximum degree. Let us choose vertices from $v_1, v_2, \dots, v_n \in V_2$ as root vertices. Then T is an m-ary tree. Let u, v be the two vertices which are at extreme distance in the spanning tree, T. That is, u, v are at maximum stretch. The distance between u and v in T is a path of length 3. Also, u and v are adjacent in G.

The fuzzy distance between every pair of vertices which are at maximum stretch is,

$$d_f^T(u,v) = 3 \times \frac{1}{2mn} = \frac{3}{2mn}$$

In G, u, v are adjacent vertices. Hence the fuzzy distance of u, v in G is

$$d_f^G(u,v) = \frac{1}{2mn}$$

Now, $d_f^T(u,v) = \frac{3}{2mn} \leq 3 \times d_f^G(u,v)$

In general, for any two vertices u, v ,

$$d_f^T(u,v) \leq 3 \times d_f^G(u,v) \text{ for } m > n$$

From Case (i) and (ii),

$$d_f^T(u,v) \leq 3 \times d_f^G(u,v)$$

In general, for any two vertices u, v ,

$$d_f^T(u,v) \leq 3 \times d_f^G(u,v)$$ ■

Theorem 3.9. The stretch factor t of a minimum tree t- spanner of a regular graph G and its RLFG G_1 are one and the same.

Proof:

Let G be a regular graph with n vertices. Let $\deg(v) = p$ for all v. Hence, $|E| = \frac{np}{2}$.

Let T be a minimum spanning tree of G generated by BFS algorithm. Let t be the stretch factor of minimum spanning tree T of G. Let u, v be the vertices which are at extreme distances in T. Let distance of u and v in G be n and distance of u and v be m in T.

$$d^T(u,v) = m, d^G(u,v) = n$$

Since t is the minimum stretch of T, $d^T(u,v) \leq t \times d^G(u,v)$

$$\Rightarrow m \leq t \times n \qquad (1)$$

Let G_1 be a RLFG of G. Let us label the vertices and edges of G using RL.

$$\sigma(v) = \frac{p}{np/2} = \frac{2}{n}, \text{ for all } v \in V, \text{ and } \mu(u,v) = \frac{1}{n} \text{ for all } (u,v) \in E(G).$$

Since G is regular, G_1 is a fuzzy regular graph by RL. For a fuzzy regular graph the edge weight $\mu(u,v)$ is equal for all the edges by RL.

Hence, $\mu(u,v) = \frac{1}{n} < \frac{2}{n} = \sigma(u) \wedge \sigma(v)$. G_1 is a fuzzy graph under RL.

Let T_1 be the minimum spanning tree of G_1, generated by BFS algorithm. Let t_1 be the stretch factor of the minimum spanning tree, T_1. We will claim that, $t_1 = t$.

As the edge weights are equal in G_1, the maximum stretch in T_1 will be between u and v as in G. The fuzzy distance between u and v in G_1 is given by,

$$d_f(u,v) = \wedge \sum \{ \bigwedge (\sigma(u),\ \sigma(v)) \times \mu(u,v)\}$$

$$d_f^{G_1}(u,v) = n \times \frac{2}{n} \times \frac{1}{n} = \frac{2}{n}$$

The fuzzy distance between *x* and *y* in T' is given by

$$d_f^{T_1}(u,v) = m \times \frac{2}{n} \times \frac{1}{n} = \frac{2m}{n^2}$$

$$d_f^{T_1}(u,v) = \frac{2m}{n^2} \leq \frac{2t \times n}{n^2}, \text{since } m \leq t \times n \ \ \text{(by (1))}$$

$$= \frac{2t}{n} = t \times \frac{2}{n} = t \times d_f^{G_1}(u,v)$$

$$\text{Thus, } d_f^{T_1}(u,v) \leq t \times d_f^{G_1}(u,v)$$

Hence, the minimum stretch factor of G_1 is t.

The stretch factor is same for G and G_1. ■

4 Ratio Labelling in General graphs

The results in section 3, presents a subclasses of graphs that fall under RLFG. This raises the question of fuzziness in arbitrary graphs under RL. So, we extended our research to analyze general graphs for admittance of fuzziness using RL. Not all, but some arbitrary graphs obey RL. This encourages to find the factors or conditions that promote fuzziness in arbitrary graphs. The degree of adjacent vertices is found to significantly influence the admittance of fuzziness under ratio labelling. In our previous work, we presented the result that ensures fuzziness in a general crisp graph. Hence, any general graph becomes a fuzzy graph under RL, for every edge (u, v) in G,

$$deg\,(u) \leq 2 \times deg\,(v),\ whenever\ deg\,(v) < deg\,(u).$$

The following theorem justifies the above statement.

Theorem 4.1.* *Let G(V, E) be any connected simple graph. Then G is a fuzzy graph under ratio labelling iff for every edge* (u,v) *of G,* $deg\,(u) \leq 2 \times deg\,(v)$ *whenever* $deg\,(v) \leq deg\,(u)$.

4.1 Minimum Tree t- Spanner of a Fuzzified Ratio Labelled General Graphs

Existence of a spanning tree in a graph ensures the connectivity between the vertices of a graph. In a weighted graph, a spanning tree with minimum weight plays a major role in applications part like communication system.

Finding tree *t*- spanner for a fuzzy graph is an emerging concept in the field of fuzzy graphs. This creates a curiosity to move on to tree *t*- spanners in RLFG. In the entire family of arbitrary graphs, a certain group of arbitrary graphs satisfies fuzziness under RL in accordance with theorem 4.1. It is worthwhile to find a spanning tree for such a group of arbitrary graphs, with minimum stretch.

***** The result is included in the paper "Algorithmic approach in Examining the Admittance of Fuzziness on General Graphs under Ratio Labelling" and is submitted to the journal "South East Asian Journal of Mathematics and Mathematical Sciences" which is under review.

To find a minimum spanner for a graph G, we need to find a spanning tree that has shortest path graph traversal. Traversal refers to the process of visiting or exploring each vertex in a graph in a systematic manner. Here, we are concern about minimum stretch not the total weight of the spanning tree. As per literature survey the BFS algorithm is the best traversing algorithm. So, we found minimum spanner of graph G through tree t-spanner by BFS algorithm without considering the edge weightage.

In order to find tree t- spanner of the general graphs G with n vertices, we categorized graph G as graphs having maximum degree $n-1, n-2, n-3$, and so on. First we considered a group of general graph G with n vertices having atleast one vertex with degree $n-1$ for determining the stretch factor t' of ratio labelled fuzzy graph G' of G. Then extended the above for a graph with maximum degree $n-2$ and highlighted all the possible situations that to be considered in determining t'. This shows the tediousness in finding t' when it is extended further for $n-3$, $n-4$ etc., thereby, showing that the problem of finding t' for an arbitrary graph is NP-complete.

Theorem 4.2. The minimum tree t – spanner problem of a ratio labelled fuzzy general graphs is NP-Complete.

Proof.

Let us prove the statement by considering a specific case, thereby analyzing the general cases of the study.
Let G be an arbitrary simple graph with n vertices and e edges with atleast one vertex of degree $n-1$.
Let G_1 be the ratio labelled fuzzy graph of G.
Let x be a vertex of degree $n-1$ in G. By ratio labelling, $\sigma(x) = \frac{\deg(x)}{e} = \frac{n-1}{e}$. Let $x_1, x_2, x_3, \dots . x_{n-1}$ be the vertices adjacent to x. The σ of $x_1, x_2, x_3, \dots . x_{n-1}$ can be maximum of $\frac{n-1}{e}$ as the graph G is simple and minimum of $\frac{n-1}{2e}$ as G_1 is fuzzy by RL. By definition of ratio labelling and theorem 4.1,

$$\frac{n-1}{2e} \leq \sigma(x_i) \leq \frac{n-1}{e}, \text{for all } i \text{ and } \mu(x, x_i) = \frac{n-1}{2e}.$$

The spanning tree, T with root vertex x has a minimum stretch. The resulting spanning tree is a $K_{1,n}$ graph, as x is adjacent to $n-1$ vertices.

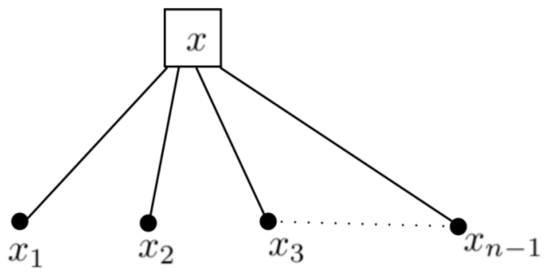


Figure 6 Spanning tree T of G

In T, the fuzzy distance between x_i and x_j ,

When $\frac{n-1}{2e} \leq \sigma(x_i)$,

$$\mathrm{d}_f^T(x_i, x_j) \geq 2 \times \frac{n-1}{2e} \times \frac{n-1}{2e} = \frac{(n-1)^2}{2e^2}$$

When $\sigma(x_i) \leq \frac{n-1}{e}$,

$$\mathrm{d}_f^T(x_i, x_j) \leq 2 \times \frac{n-1}{e} \times \frac{n-1}{2e} = \frac{(n-1)^2}{e^2}$$

Hence, $\frac{(n-1)^2}{2e^2} \leq d_f^T(x_i, x_j) \leq \frac{(n-1)^2}{e^2}$ (1)

If x_i and x_j are adjacent in G,

For $\frac{n-1}{2e} \leq \sigma(x_i)$,

$$d_f^{G_1}(x_i, x_j) \geq \frac{n-1}{2e} \times \frac{n-1}{4e} = \frac{(n-1)^2}{8e^2}$$

For $\sigma(x_i) \leq \frac{n-1}{e}$,

$$d_f^{G_1}(x_i, x_j) \leq \frac{n-1}{e} \times \frac{n-1}{2e} = \frac{1}{2}\frac{(n-1)^2}{e^2}$$

Hence, $\frac{(n-1)^2}{8e^2} \leq d_f^{G_1}(x_i, x_j) \leq \frac{1}{2}\frac{(n-1)^2}{e^2}$

$$\frac{(n-1)^2}{2e^2} \leq 4\, d_f^{G_1}(x_i, x_j) \leq 2\,\frac{(n-1)^2}{e^2} < \frac{(n-1)^2}{e^2} \qquad (2)$$

From (1) and (2),

$$\mathrm{d}_f^T(x_i, x_j) = 4 \times d_f^{G_1}(x_i, x_j) \qquad (3)$$

Suppose x_i and x_j are not adjacent,

$$\frac{(n-1)^2}{2e^2} \leq d_f^{G_1}(x_i, x_j) \leq \frac{(n-1)^2}{e^2}$$

$$d_f^T(x_i, x_j) = 1 \times d_f^{G_1}(x_i, x_j) \qquad (4)$$

From (3) and (4), $1 \leq t' \leq 4$

Hence, the stretch factor for a ratio labelled arbitrary graph G_1 on n vertices, with a vertex having maximum degree $n-1$ has lower bound as 1 and upper bound as 4.

The above result can be extended for a graph with n vertices having a vertex with maximum degree $n-2$. Let the vertex x be adjacent to $n-2$ vertices, $x_1, x_2, \ldots, x_{n-2}$ and non-adjacent to y in G. This y is adjacent with any number of the vertices of $x_1, x_2, \ldots, x_{n-2}$.

In G_1,

$$\sigma(x) = \frac{n-2}{e}$$

$$\frac{n-2}{2e} \leq \sigma(x_i) \leq \frac{n-2}{e} \ for\ all\ i$$

$$\mu(x, x_i) = \frac{n-2}{2e}$$

and

$$\frac{n-2}{4e} \leq \mu(x_i, x_j) \leq \frac{n-2}{2e} for\ i \neq j.$$

The vertex y is adjacent to some of the vertices of $x_1, x_2, \dots, x_{n-2}$in G. In G_1,

$$\frac{n-2}{4e} \leq \sigma(y) \leq \frac{n-2}{e},$$

depending on the adjacency between the vertices $x_1, x_2, \dots, x_{n-2}$ and y

$$\frac{n-2}{4e} \leq \mu(x_i, y) \leq \frac{n-2}{2e}$$

The stretch t' of G_1 is determined by the vertices $x_1, x_2, \dots, x_{n-2}$and y. In T, suppose the minimum stretch lies between x_i and y.

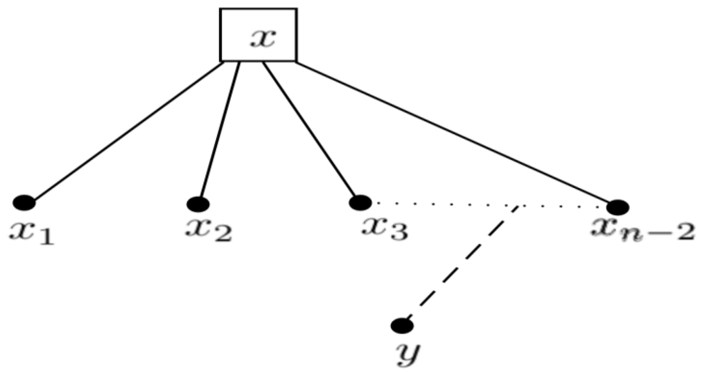


Figure 7 Spanning tree of G with x having degree $n-1$

Now, when $\frac{n-2}{2e} \leq \sigma(x_i)$ and $\frac{n-2}{4e} \leq \sigma(y)$,

$$d_f^T(x_i, y) = d_f^T(x_i, x) + d_f^T(x, x_j) + d_f^T(x_j, y)$$

$$d_f^T(x_i, y) \geq \frac{n-2}{2e} \times \frac{n-2}{2e} + \frac{n-2}{2e} \times \frac{n-2}{2e} + \frac{n-2}{4e} \times \frac{n-2}{4e}$$

$$= \frac{(n-2)^2}{(2e)^2}\left[1+1+\frac{1}{4}\right] = \frac{9}{16}\frac{(n-2)^2}{(e)^2}$$

$$Hence, d_f^T(x_i, y) \geq \frac{9}{16}\frac{(n-2)^2}{(e)^2}$$

Again, when $\sigma(x_i) \leq \frac{n-2}{e}$ and $\sigma(y) \leq \frac{n-2}{e}$

$$d_f^T(x_i, y) \leq \frac{n-2}{e} \times \frac{n-2}{2e} + \frac{n-2}{e} \times \frac{n-2}{2e} + \frac{n-2}{e} \times \frac{n-2}{2e}$$

$$= \frac{(n-2)^2}{2(e)^2}\,[1+1+1] \;=\; \frac{3}{2}\frac{(n-2)^2}{(e)^2}$$

$$Hence, d_f^T(x_i, y) \leq \frac{3}{2}\frac{(n-2)^2}{(e)^2}$$

$$\frac{9}{16}\frac{(n-2)^2}{(e)^2} \leq \; d_f^T(x_i, y) \leq \frac{3}{2}\frac{(n-2)^2}{(e)^2} \qquad (5)$$

Now, the fuzzy distance between x_i, y in G_1 is found under two cases when, (i) x_i, y are adjacent in , (ii) x_i, y not adjacent in G

Case (i)

Let x_i, y are adjacent in G_1

For, $\frac{n-2}{2e} \leq \sigma(x_i)$ and $\frac{n-2}{4e} \leq \sigma(y)$

$$d_f^{G_1}(x_i, y) \geq \frac{n-2}{4e} \times \frac{n-2}{4e} = \frac{(n-2)^2}{16(e)^2}$$

For $\sigma(x_i) \leq \frac{n-2}{e}$ and $\sigma(y) \leq \frac{n-2}{e}$

$$d_f^{G_1}(x_i, y) \leq \frac{n-2}{e} \times \frac{n-2}{2e} = \frac{(n-2)^2}{2(e)^2}$$

$$\frac{(n-2)^2}{16(e)^2} \leq \; d_f^{G_1}(x_i, y) \leq \frac{(n-2)^2}{2(e)^2}$$

$$\frac{9}{16}\frac{(n-2)^2}{(e)^2} \leq 9\, d_f^{G_1}(x_i, y) \leq \frac{9(n-2)^2}{2(e)^2} < \frac{3(n-2)^2}{2(e)^2} \qquad (6)$$

$$From\ (5)\ and\ (6), d_f^T(x_i, y) = 9d_f^{G_1}(x_i, y) \qquad (7)$$

Case (ii).

Let x_i, y not adjacent in G. Then the length of the path between x_i and y either be 2 or 3. The path may be

(a) x_i, x_{i+1}, y (or)

(b) x_i, x_{i+1}, x_{i+2}, y (or)

(c) x_i, x, x_j, y

Case(a)

When the path between x_i and y is x_i, x_{i+1}, y, then

$$d_f^{G_1}(x_i, y) \geq \frac{n-2}{2e} \times \frac{n-2}{4e} + \frac{n-2}{4e} \times \frac{n-2}{4e} = \frac{3}{16}\frac{(n-2)^2}{(e)^2}$$

and

$$d_f^{G_1}(x_i, y) \leq \frac{n-2}{e} \times \frac{n-2}{2e} + \frac{n-2}{e} \times \frac{n-2}{2e} = \frac{(n-2)^2}{(e)^2}$$

Hence

$$\frac{3(n-2)^2}{16(e)^2} \leq d_f^{G_1}(x_i, y) \leq \frac{(n-2)^2}{(e)^2} \tag{8}$$

Case (b)

When the path between x_i and y is x_i, x_{i+1}, x_{i+2}, y then

$d_f^{G_1}(x_i, y) \geq \frac{n-2}{2e} \times \frac{n-2}{4e} + \frac{n-2}{2e} \times \frac{n-2}{4e} + \frac{n-2}{4e} \times \frac{n-2}{4e} = \frac{5}{16}\frac{(n-2)^2}{(e)^2}$ and

$d_f^{G_1}(x_i, y) \leq \frac{n-2}{e} \times \frac{n-2}{2e} + \frac{n-2}{e} \times \frac{n-2}{2e} + \frac{n-2}{e} \times \frac{n-2}{2e} = 3\,\frac{(n-2)^2}{2(e)^2}$

Hence

$$\frac{5(n-2)^2}{16(e)^2} \leq d_f^{G_1}(x_i, y) \leq \frac{3(n-2)^2}{2(e)^2} \tag{9}$$

From (8) and (9),

$$\frac{3(n-2)^2}{16(e)^2} \leq d_f^{G_1}(x_i, y) \leq \frac{3(n-2)^2}{2(e)^2}$$

$$\frac{9(n-2)^2}{16(e)^2} \leq 3\, d_f^{G_1}(x_i, y) \leq \frac{9(n-2)^2}{2(e)^2} \leq \frac{3(n-2)^2}{2(e)^2} \tag{10}$$

From (6) and (10) $\quad d_f^{T}(x_i, y) = 3\, d_f^{G_1}(x_i, y)$ (11)

From (7) and (11) $\quad 3 \leq t' \leq 9$ (12)

Case (c)

When the path between x_i and y is x_i, x, x_j, y, which gives the same fuzzy distance between x_i, y as in

T.

$$\text{i.e.} \frac{9}{16}\frac{(n-2)^2}{(e)^2} \leq d_f^G(x_i, y) \leq \frac{3}{2}\frac{(n-2)^2}{(e)^2}$$

Hence from (5) $$d_f^T(x_i, y) = d_f^G(x_i, y) \quad (13)$$

From (12) and (13) $1 \leq t' \leq 9$

The stretch factor t' lies between 1 and 9 when G has a vertex with maximum degree $n-2$.

Extending the proof for a graph G with n vertices and with a vertex having maximum degree $n-k$, for some k, finding the stretch t' of G_1 depends on the adjacency between the $k-1$ vertices, y_1, $y_2, y_3, \ldots\ldots, y_{k-1}$ that are not adjacent to the vertex x and those vertices that are adjacent to x, say $x_1, x_2, \ldots\ldots, x_{n-k}$.

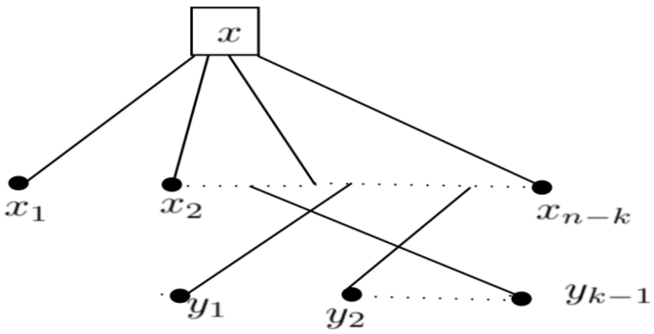


Figure 8 Spanning tree of G with x having degree $n-k$

Without the knowledge of adjacency between $\boldsymbol{x_1, x_2, \ldots\ldots, x_{n-k}}$, $\boldsymbol{y_2, y_3, \ldots\ldots, y_{k-1}}$ the stretch $\boldsymbol{t'}$ cannot be determined. Thus in the above particular case, finding the stretch $\boldsymbol{t'}$ is possible only upto some extend. Even for a graph with maximum degree $\boldsymbol{n-4}$, there are plenty of cases to discuss regarding the adjacency between $\boldsymbol{y_1, y_2, y_3}$ and $\boldsymbol{y_2, y_3, \ldots\ldots, y_{n-4}}$. So the problem of finding $\boldsymbol{t'}$ is more complicated in this particular case and it is not possible to define a stretch $\boldsymbol{t'}$ for a general $\boldsymbol{n-k}$.
When this is extended for a general graph that does not fall under this particular case that we discussed above, the concept of determining t' for a RLFG G_1 is not possible as adjacency between the n vertices are not known.
On all other general graphs with n vertices whose vertices do not have maximum degree as $-1, n-2, \ldots n-k$, the stretch factor depends on the choice of the root vertex. Thus, the stretch t' varies with the choice of root vertex of the spanning tree. Determining the stretch factor t' of a minimum spanning tree of a RLFG is ambiguous, without knowing the degree of the vertices and their adjacency. Generalization of this concept to an arbitrary graph is not possible, as the stretch factor depends on the number of vertices and their adjacency. Therefore, for any general graph, finding a tree t-spanner of a RLFG is quite impossible, and the problem is NP-complete. ∎

5 Application of minimum tree spanner with Link Prediction

As an application, we analyzed the communication bond between a group of people in a social media by considering the people as vertices and the communication between them as edges. The graph was examined for fuzziness using RL. Depending on the time, the interaction between the individuals varies, and hence the graph found to be either a non-fuzzy graph or fuzzy graph under RL. When there is a good communication between the group members, the graph admits fuzziness under RL, which ensures the

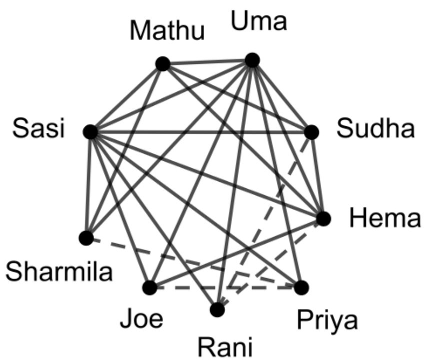
strong bond between them. The graph $G$for a particular group of friends is given in Figure 9. The graph G is labelled using RL and it is found to be fuzzy [2]. In this paper, we extend this application to find the spanner of the RLFG G. The vertices and edges are labelled using RL as, $\sigma(Sasi) = \frac{8}{24} = \sigma(Uma)$; $\sigma(Mathu) = \sigma(Sudha) = \frac{5}{24}$;
$\sigma(Hema) = \frac{6}{24}$; $\sigma(Sharmila) = \sigma(Joe) = \sigma(Priya) = \sigma(Rani) = \frac{4}{24}$.

For any $x, x \neq Sasi, Uma$

$\mu(Sasi, x) = \mu(Uma, x) = \frac{8}{48} = \frac{1}{6}$; $\mu(Hema, x) = \frac{6}{48} = \frac{1}{8}$; $\mu(Mathu, y) = \mu(Sudha, y) = \frac{5}{48}$, where $y \neq Sasi, Uma, Hema$; $\mu(u, v) = \frac{4}{48} = \frac{1}{12}$, where $u, v = Priya, Sharmila, Joe$ and $u \neq v$. For all $u, v \in G$, $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. Hence, $G$is a fuzzy graph due to RL.
The minimum tree t- spanner for Figure 9 was constructed with a spanning tree having the vertex Sasi as root vertex, the one with maximum degree. The spanning tree T for the graph $G$is given in Figure 10.

From Figure 9, 10, $d_f^T(u, v) \leq 4d_f^G(u, v)$.

The spanner $t = 4$ is attained between the vertices $(Sharmila, Priya)$ and $(Priya, Joe)$ as $d_f^T(Sharmila, Priya) = d_f^T(Priya, Joe) = \frac{32}{24^2}$ and $d_f^G(Sharmila, Priya) = d_f^G(Priya, Joe) = \frac{8}{24^2}$ and who are less communicative. Thus, fuzzy tree-spanner identifies the less communicative people.



Graph G representing Communication Bond
Figure 9

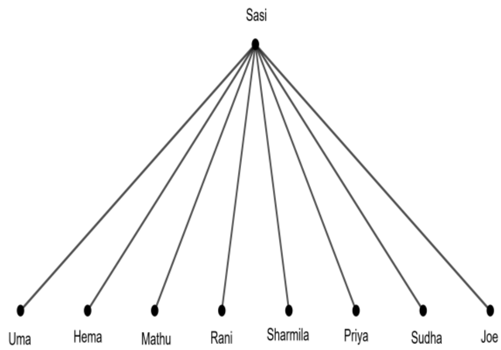


Figure 10 Spanning Tree T of G

Also we extended this problem to predict the linkage between two non-adjacent vertices. The link Prediction problem helps to identify whether a pair of non-adjacent vertices will become adjacent in the future. RSM is a link prediction method which is used to predict the linkage between two non-adjacent vertices [28]. The parameter used in link prediction method, RSM is called as familiarity index fam(i).

$fam(i) = min[\boldsymbol{\mu(u, w_i)}, \boldsymbol{\mu(v, w_i)}]$ where $\boldsymbol{w_i}$ is a common neighbour of u and v.

Link prediction score is obtained using, $\boldsymbol{S_{uv}} = \sum_{i=1}^{n} \frac{fam(i)}{n}$.

S.No.	**Non Adjacent Edges (a, b) in G**	**LP**
1	(Mathu, Priya)	0.139

2	(Mathu, Rani)	0.14
3	(Mathu, Joe)	0.15
4	(Sudha, Priya)	0.11
5	(Sudha, Joe)	0.15
6	(Sudha, Sharmila)	0.146
7	(Hema, Priya)	0.138
8	(Hema, Sharmila)	0.138
9	(Priya, Rani)	0.167
10	(Rani, Joe)	0.15
11	(Rani,Sharmila)	0.167
12	(Joe, Sharmila)	0.21

Table 1.

Further the people who do not communicate with one another are listed in Table 1. The possibility for communication is found by RSM method in Link Prediction by calculating familiarity index. One with the maximum score shows the greater chance for communication. From the Table 1 it is found that Joe and Sharmila has more chance to communicate in future than others.

Conclusion.

The paper examines the fuzzy tree t-spanner of RLFGs. It highlights the minimum spanner of RLFG. Notably, the ratio labelling of a graph does not impact the stretch factors of regular graph. The stretch factors of a tree t – spanner of ratio-labelled regular fuzzy graphs are identical to those regular crisp graphs. The chapter also analyzed the minimum tree t- spanner of ratio labelled general graphs. For general graphs finding tree t-spanner is NP-Complete. An application representing the relationship bond between the people was analyzed with RSM link prediction method. Examining fuzzy tree t-spanner for ratio labelled neural networks is under proposal.

References

[1] Althofer I, Das G., Dobkin D., Joseph D., Soares J., On Sparse Spanners of Weighted Graphs, Discrete Comput. Geom., 9, 81-100 (1993).

[2] Amutha, A., Mathu Pritha, R., Fuzzification of Crisp and its Implications, Journal of Tianjin University Science and Technology.: 58(08), 315-334(2025). .

[3] Amutha, A., Jebakiruba, C., Davamani Christober, M., An Effective Algorithm to Enumerate Spectrum of Laplacian Matrix for Complete Fuzzy Graphs, Bhateja, V., Satapathy, S.C., Traviesco-Gonzalez, C.M., Aradhya, V.N.M.(eds) Data Engineering and Intelligent Computing. Advances in Intelligent Systems and Computing, 1407, Springer, Singapore, *(2021)*.

[4] Amutha A, Mathu Pritha R., Fuzziness on Interconnection Networks Under Ratio Labelling, Smart Innov. Syst. Technol., 266, 165-173(2022). https://doi.org/10. 1007/978-981-16-6624-7_17

[5] Awerbuch B., Baratz A., Peleg D., Efficient Broadcast and Light-Weight Spanners, Manuscript, (1992).

[6] Blue M., Bush B., Puckett J., "Unified approach to fuzzy graph problems", Fuzzy Sets and Systems, 125, 355-368(2002).https://doi.org/10.1016/S0165-0114(01)00011-2

[7] Brandstadt A., F.F.Dragan F.F., H.O. Le , V.B.L.E., Tree Spanners on Chordal Graphs: Complexity, Algorithms, Open Problems, Algorithms and Computation, 163-174, (2002)

[8] Brankovic M, Gudmundsson J, Renssen AV., Local Routing in a Tree Metric 1-Spanner, J. Comb. Optim.,44, 2642-2660(2022). https://doi.org/10.1007/s10878-021-00784-4

[9] Couto F, Cunha LF, Daniel Juventude D, Santiago L., Strategies for Generating tree spanners, Algorithms, heuristics and optimal graph classes. Inf. Process. Lett., 177, 106265, (2022).http://doi.org/10.1016/j.ipl.2022.106265

[10] Couto F, Cunha L, Ferraz D. Simpler and Efficient Characterizations of Tree t-Spanners for Graphs with few P4's and (k, l)-graphs. Cornell University – arXiv. *(2022)*. https: //doi.org/10.48550/arXiv.2208.14309

[11] David Peleg and Alejandro A Schaffer., "Graph spanners, Journal of Graph Theory," *13(1):99-116(1989)*.

[12] David Peleg, Ullaman J.D., An Optimal Synchronizer for the Hypercube, Proceedings of the Sixth ACM Symposium on Principles of Distributed Computing, *pp.77-85,(1987)*.

[13] Dey A, Son LH, Pal A, Long HV., "Fuzzy minimum spanning tree with interval type 2 fuzzy arc length: formulation and a new genetic algorithm. Soft Comput.,"*;*24,3963-74,2020 https://doi.org/10.1007/s00500-019-04166-1.

[14] Deshpande A A, Chaudhari O K., "Fuzzy approach to compare a minimal spanning tree problem by using various algorithms.," *AFM,* 15(1),47-58, 2020 http://www. ripublication.com/afm.htm.

[15] Gomez R, Miyazawa F K, Wakabayshi Y., "Tree 3-Spanners on generalized prisms of graphs," *LNCS,* 13568, 557-73,2022. https://doi.org/10.1007/978-3-031-20624-5_34

[16] Ioannis Papoutsakis.,"Tree Spanners of Bounded Degree Graphs", Discrete Applied Mathematics,: 236, 395-407(2018).

[17] Kauffman, A.,"Introduction a La Theorie Des Sous-Emsembles Flous", *Vol. I, Masson et cie ,(1973)*.

[18] Liestman A.L, Shermer T, "Grid Spanners, Networks," *23:123-133(1993)*.

[19] Madhumangal Pal., Sovan Samanta., Ganesh Ghorai., "Modern Trends in Fuzzy Graph Theory" *Springer, Singapore, 2020*.

[20] Mary F R, Mohanaselvi S, Broumi S., "A solution approach to minimum spanning tree problem under fermatean fuzzy environment. Bull. Electr. Eng., Inform.", 12(3), 1738-46, 2023 https://doi.org/10.11591/eei.v12i3.4794

[21] Mathew Varkey, T K.,Sreena, T D." Fuzzification of Some Path Related Graphs", Interna- tional Journal of Advanced Research in Engineering and Applied Sciences,*vol. 6, no. 10, (2017)*.

[22] Mathew Varkey, T K., Sreena, T D., "Fuzzification of Generalised Petersen Graph", Interna-

tional journal of Mathematics Trends and Technology, *54(2):133-137(2018)*.

[23] Mohanta K, Day A, Debnath NC, Pal A., "An algorithmic approach for finding minimum spanning tree in inntuitionistic fuzzy graph.," *CAINE 2019,* 63, 140-9, 2019. http://doi. org/10.29007/sx3s.

[24] Myna R., "Application of Fuzzy Graph in Traffic", International Journal of Scientific & Engineering Research, *Vol 6, Issue 2(2015)*

[25] Nagoor Gani,A., Rajalaxmi(a) Subahashini, D.," Properties of Fuzzy Labeling Graph", Applied Mathematical Sciences, *vol. 6, no. 70, 3461- 3466,(2012)*.

[26] Richards D, Liestman A.L., ""Degree Constrained pyramid spanners", Parallel Distributed Comput., *25:1-6(1995)*.

[27] Rosenfeld, A.," Fuzzy Graphs,in Fuzzy Sets and their Applications to Cognitive and Deci- sion Process", *M.,Eds. Academic Press,New York, 77- 95, (1975)* .

[28] Rupkumar Mahapatra, Sovan Samanta, Madhumangal Pal, Qin Xin, RSM index, A new way of Link Prediction in social Networks, Journal of Intelligent & Fuzzy Systems, vol. 35(5), pp-1-15.

[29] Sinthamani P., "An Application of Fuzzy Graph in Accidental Prone Zone to Reduce the Traffic Congestion", Malaya Journal of Mathematik, *vol. 9,n0. 1, 378-384(2021)*.

[30] Zadeh, L.,"Fuzzy sets", Information and Control, *8, pp. 338-353(1965)*.