

An Approach to Solution of Fractional Kinetic Equations Associated with Hyper-Bessel Function and k-Mittag-Leffler Function

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Abstract

In this present work, expressed the generalized fractional kinetic equations associated with Hyper-Bessel function and their fractional derivatives. Additionally, solutions are derived in terms of the Mittag-Leffler function by employing the Laplace transform approach. Furthermore, a graphical depiction of the solutions is offered to discuss particular cases of fractional kinetic equations and show how they behave.

Key words: Generalized fractional kinetic equation, Hyper-Bessel function, Fractional derivative, Laplace Transform, Mittag-Leffler function.

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1. Introduction

Due to the importance of Hyper-Bessel functions, which appear in the various fields of applied and pure mathematical frameworks. The significant applications of Hyper-Bessel are in science and engineering, and it is widely used in the solution of fractional-order differential equations.

Over the past few decades, fractional differential equations (FKE) have become increasingly important in important fields of science and engineering. These equations have numerous important uses in the social sciences, as well as in science and engineering, such as image processing, dynamical systems, and control systems.

Because of the great significance of kinetic equations and to enhance their scientific and mathematical applications, Haubold and Mathai [13] established a generalization of the FKE Concerning the rate of change of the reaction $N=N(t)$, rate of destruction $d = d(N)$ and rate of production $p=p(N)$ as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t) \quad \dots(1.1)$$

where $N = (N_t)$ denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

Additionally, the particular case of (1.1) for spatial fluctuations or inhomogeneities in the quantity N_t are ignored, which is represented as

$$\frac{dN_i}{dt} = -c_i N_i(t) \quad \dots(1.2)$$

with the initial condition $N_i(t = 0) = N_0$, the number of densities of a species i at the time $t = 0$, $c_i > 0$.

Dropping the index i , and solving (1.2) they get

$$N(t) - N_0 = c_0 D_t^{-1} N(t) \tag{1.3}$$

On replacing Riemann-Liouville fractional integral operator present on the right-hand side of (1.3) by the fractional integral operator ${}_0D_t^{-\nu}$ studied by Miller and Ross [17], a generalized standard fractional kinetic equation was obtained as

$$N(t) - N_0 = c^\nu {}_0D_t^{-\nu} N(t) \tag{1.4}$$

where ${}_0D_t^{-\nu}$ is the fractional integral operator, defined as

$${}_0D_t^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (t-u)^{\nu-1} f(u) du, x > 0, Re(\nu) > 0 \tag{1.5}$$

Additionally, Saxena and Kalla [20] investigated and described the generalization of the FKE as

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t), Re(\nu) > 0 \tag{1.6}$$

where $N(t)$ represents the number density of species at the time t , N_0 represents the number of densities at the time $t = 0$, c is the constant, and $f(t) \in L(0, \infty)$.

Furthermore, several researchers investigated the extension of FKEs associated with special functions and produced several intriguing findings that are quite significant like Agarwal and Bhargava [1] Agrawal et al. [2], Chaurasia and Pandey [9], Gupta and Parihar [11], Gupta et al. [12], Haubold and Mathai [13], Kumar et al. [14], Nisar et al. [18], Saichev and Zaslavsky [22], Saxena and Kalla [23], Saxena et al. [24], Singh et al. [25], Tariboon et al. [30] and Zaslavsky [32] etc. are some of them.

Furthermore, we offer a generalized variant of the fractional kinetic equation involving the Hyper-Bessel function and its fractional derivatives to illustrate the utility and importance of the equation in certain astrophysical problems. Afterward, we use the Laplace Transform to find a solution.

The generalized form of the Hyper-Bessel function [6] is presented as

$$\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi) = \left(\frac{\xi}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n} \tilde{\mathfrak{J}}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi)$$

where

$$\tilde{\mathfrak{J}}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{\xi}{n+1}\right)^{r(n+1)} \tag{1.7}$$

where $\xi, \tau_i \in \mathbb{C}, Re(\tau_i + 1) > 0, |\xi| < \infty, i = 1, 2, \dots, n$. For more details, one can refer [3,4,5,6,21].

Fractional Order Derivative of Hyper-Bessel function

The fractional order derivative [19,20] of order λ of the function $f(t) = t^\beta$ is given by

$$D^\lambda t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \lambda + 1)} t^{\beta - \lambda}; Re(\beta) > -1, 0 < Re(\lambda) < 1, t > 0 \tag{1.8}$$

So, in view of (1.7) and (1.8) we have

$$\begin{aligned} & {}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi) \right) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\times \frac{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + 1)}{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) - \lambda + 1)} (\xi)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) - \lambda} \end{aligned} \tag{1.9}$$

The Laplace Transform, defined by Pierre-Simon Laplace [26] and represented as

$$L[f(t); s] = F(s) = \int_0^\infty f(t)e^{-ts} dt \quad ; t > 0, Re(s) > 0 \quad \dots(1.10)$$

Laplace Transform of Hyper-Bessel function

$$\begin{aligned} &\mathcal{L}\left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi)\right) \\ &= \sum_{r=0}^\infty \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r!} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\times (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + \lambda)} \end{aligned} \quad \dots(1.11)$$

Proof: By using the definition of Laplace Transform and Hyper-Bessel function

$$\begin{aligned} &\mathcal{L}\left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi)\right) \\ &= \int_0^\infty e^{-\xi s} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \sum_{r=0}^\infty \frac{(-1)^r}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r!} (\xi)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} d\xi \end{aligned}$$

by changing the order of summation and integral (which is permissible under the given conditions), we get

$$\begin{aligned} &\mathcal{L}\left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi); s\right) \\ &= \sum_{r=0}^\infty \frac{(-1)^r}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r!} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \int_0^\infty e^{-\xi s} (\xi)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} dt \\ &= \sum_{r=0}^\infty \frac{(-1)^r}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r!} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \times \frac{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + 1)}{(s)^{(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + 1)}} \end{aligned}$$

Laplace Transform of Fractional Derivative of Hyper-Bessel function

Based on (1.9) and (1.11), we can readily conclude that

$$\begin{aligned} &\mathcal{L}\left[{}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(\xi)\right); s\right] \\ &= \sum_{r=0}^\infty \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r!} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\times (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) - \lambda + 1)} \end{aligned} \quad \dots(1.12)$$

In the proposed work, we find the results in terms of Mittag-Leffler function [30] defined as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha) > 0, Re(\beta) > 0, \alpha, \beta \in \mathbb{C} \quad \dots(1.13)$$

Fractional Derivative of composition of Hyper-Bessel function and generalized k-Mittag-Leffler function

$$\begin{aligned} &{}_0D_t^\lambda \left\{ \mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{k, \zeta, \tau}^{\xi, \eta}(t) \right\} \\ &= \sum_{r=0}^\infty \sum_{m=0}^\infty \frac{(-1)^r (\xi)_{m\eta, k}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1)r! k! \Gamma(k(m\zeta + \tau))} \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\times \frac{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) - \lambda + m + 1)} (t)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda} \end{aligned} \quad \dots(1.14)$$

Laplace Transform of Fractional Derivative of composition of Hyper-Bessel function and k-Mittag-Leffler function

$$\begin{aligned}
 & L \left[{}_0D_t^\lambda \left\{ \mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{k, \zeta, \tau}^{\xi, \eta}(t) \right\} \right] \\
 &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\
 &\times (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1)}
 \end{aligned} \tag{1.15}$$

2. Main Results

Theorem 1: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, r \in \mathbb{R}, \eta \in (0, 1) \cup \mathbb{N}, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\tau) > 0$ and μ is an arbitrary parameter, then the solution of the FKE

$$N(t) - N_0 \left\{ \mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{k, \zeta, \tau}^{\xi, \eta}(t) \right\} = -c^v {}_0D_t^{-v} N(t) \tag{2.1}$$

is given by

$$\begin{aligned}
 & N(t) \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! (n+1)^{r(n+1)}} \\
 &\quad \times E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}(-c^v t^v)
 \end{aligned} \tag{2.2}$$

Proof: By the approach of Laplace Transform on (2.1)

$$\begin{aligned}
 & N(s) \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\
 &\times \frac{1}{s^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}} \times (1 + c^v s^{-v})^{-1} \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{1}{n+1} \right)^{r(n+1)} \\
 &\quad \times \frac{1}{s^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}} \sum_{\alpha=0}^{\infty} (-c^v s^{-v})^\alpha \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{1}{n+1} \right)^{r(n+1)} \\
 &\quad \times \sum_{\alpha=0}^{\infty} (-c^v)^\alpha s^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + v\alpha + 1)}
 \end{aligned} \tag{2.3}$$

Now taking inverse Laplace Transform of (2.3)

$$\begin{aligned}
 & N(t) \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! (n+1)^{r(n+1)}} \\
 &\times \sum_{\alpha=0}^{\infty} \frac{(-c^v t^v)^\alpha}{\Gamma(v\alpha + \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)} \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! (n+1)^{r(n+1)}}
 \end{aligned}$$

$$\times E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}(-c^v t^v)$$

Graphical Numerical Interpretation of equation 2.2

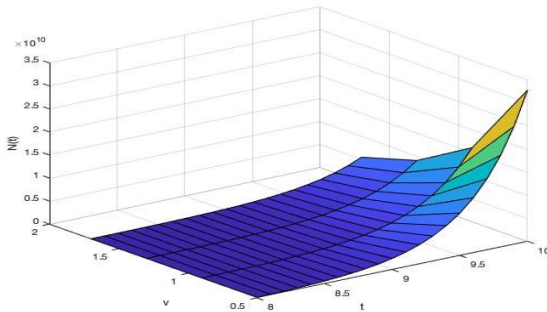


Figure-1(A)

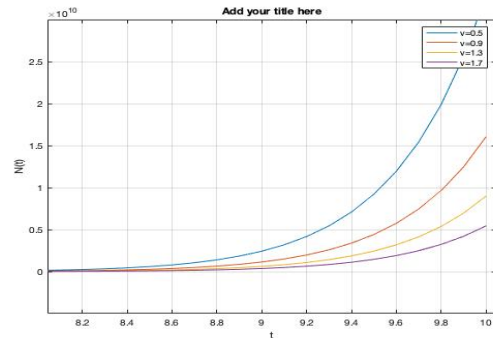


Figure-1(B)

Table 1: The values of $N(t)$ in GFKE in equation 2.2 at several values of t and v

t	Fix $v = 0.5$	Fix $v = 0.9$	Fix $v = 1.3$	Fix $v = 1.7$
	$N(t)$	$N(t)$	$N(t)$	$N(t)$
2	1.287E-48	2.6341E-49	6.00375E-50	1.39758E-50
2.8	1.80233E-47	4.12551E-48	1.06545E-48	2.82686E-49
3.6	1.32906E-46	3.30036E-47	9.34089E-48	2.7298E-48
4.4	6.69385E-46	1.77143E-46	5.38711E-47	1.69918E-47
5.2	2.61812E-45	7.29786E-46	2.35385E-46	7.90579E-47
6	8.55346E-45	2.49077E-45	8.4424E-46	2.99047E-46
6.8	2.44256E-44	7.38541E-45	2.61261E-45	9.69021E-46
7.6	6.28358E-44	1.96353E-44	7.21113E-45	2.78498E-45
8.4	1.48727E-43	4.7853E-44	1.8168E-44	7.27365E-45
9.2	3.28885E-43	1.08629E-43	4.24901E-44	1.75703E-44
10	6.87313E-43	2.32464E-43	9.3414E-44	3.97771E-44
2	1.287E-48	2.6341E-49	6.00375E-50	1.39758E-50
2.8	1.80233E-47	4.12551E-48	1.06545E-48	2.82686E-49
3.6	1.32906E-46	3.30036E-47	9.34089E-48	2.7298E-48
4.4	6.69385E-46	1.77143E-46	5.38711E-47	1.69918E-47
5.2	2.61812E-45	7.29786E-46	2.35385E-46	7.90579E-47
6	8.55346E-45	2.49077E-45	8.4424E-46	2.99047E-46
6.8	2.44256E-44	7.38541E-45	2.61261E-45	9.69021E-46
7.6	6.28358E-44	1.96353E-44	7.21113E-45	2.78498E-45
8.4	1.48727E-43	4.7853E-44	1.8168E-44	7.27365E-45
9.2	3.28885E-43	1.08629E-43	4.24901E-44	1.75703E-44

Theorem-2: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, d \neq c, r \in \mathbb{R}, \eta \in (0, 1) \cup \mathbb{N}, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\tau) > 0$. Then the solution of the FKE

$$N(t) - N_0 \left\{ \mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^v t^v) E_{k, \zeta, \tau}^{\xi, \eta}(c^v t^v) \right\} = -d^v {}_0 D_t^{-v} N(t) \tag{2.4}$$

is given as

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \times \left(\frac{ct}{n+1} \right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times E_{v, (v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}(-d^v t^v)$$

...(2.5)

Proof: By the approach of Laplace Transform on (2.4)

$$\begin{aligned}
 &N(s) \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta,k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \\
 &\times \frac{1}{s^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1}} (1 + d^v s^{-v})^{-1} \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta,k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + v(r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \\
 &\times \sum_{\alpha=0}^{\infty} (-d^v s^{-v})^\alpha s^{-(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)} \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta,k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \\
 &\times \sum_{\alpha=0}^{\infty} (-d^v)^\alpha s^{-(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + v\alpha + 1)}
 \end{aligned}$$

Now taking inverse Laplace Transform, we have

$$\begin{aligned}
 &N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta,k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\
 &\times \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times \sum_{\alpha=0}^{\infty} (-d^v)^\alpha \frac{(t)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + v\alpha}}{\Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + v\alpha + 1)} \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta,k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\
 &\quad \left(\frac{ct}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times E_{v, (v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}(-d^v t^v)
 \end{aligned}$$

Graphical Numerical Interpretation of equation 2.5

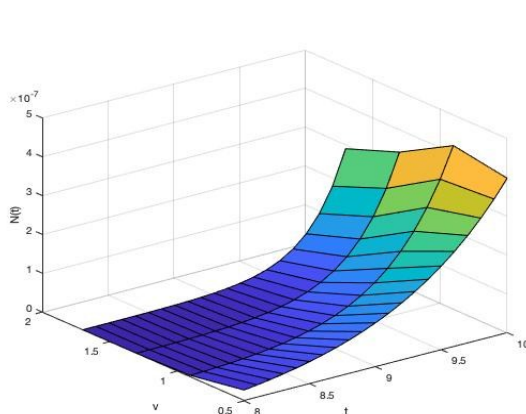


Figure-2 (A)

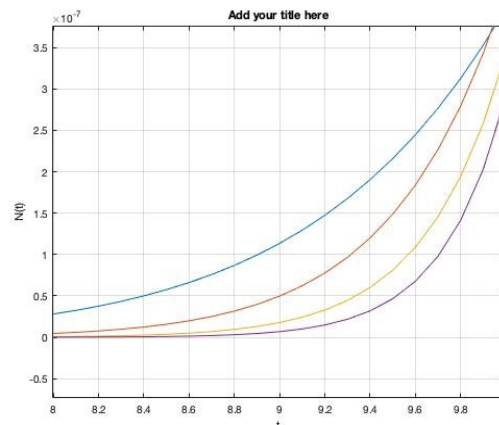


Figure-2 (B)

Table 2: The values of $N(t)$ in GFKE in equation 2.5 at several values of t and v

t	Fix $v = 0.5$	Fix $v = 0.9$	Fix $v = 1.3$	Fix $v = 1.7$
		$N(t)$	$N(t)$	$N(t)$

8	2.80177E-08	4.62776E-09	6.6085E-10	1.02598E-10
8.1	3.24649E-08	5.94129E-09	9.34277E-10	1.5949E-10
8.2	3.7552E-08	7.6054E-09	1.3155E-09	2.46628E-10
8.3	4.33615E-08	9.70788E-09	1.84496E-09	3.79421E-10
8.4	4.99858E-08	1.23571E-08	2.57755E-09	5.80796E-10
8.5	5.75276E-08	1.56866E-08	3.58751E-09	8.84712E-10
8.6	6.61015E-08	1.98603E-08	4.97486E-09	1.34124E-09
8.7	7.58343E-08	2.50792E-08	6.87398E-09	2.02387E-09
8.8	8.68672E-08	3.15891E-08	9.46476E-09	3.04005E-09
8.9	9.93564E-08	3.96902E-08	1.29874E-08	4.54615E-09
9	1.13475E-07	4.97477E-08	1.77612E-08	6.76885E-09
9.1	1.29415E-07	6.22057E-08	2.42103E-08	1.00354E-08
9.2	1.47387E-07	7.76025E-08	3.28951E-08	1.48165E-08
9.3	1.67624E-07	9.659E-08	4.45551E-08	2.17863E-08
9.4	1.90385E-07	1.19955E-07	6.01626E-08	3.1907E-08
9.5	2.15953E-07	1.48647E-07	8.09929E-08	4.65468E-08
9.6	2.44638E-07	1.83808E-07	1.08714E-07	6.76438E-08
9.7	2.76785E-07	2.26808E-07	1.45501E-07	9.7934E-08
9.8	3.12769E-07	2.79294E-07	1.94184E-07	1.41266E-07
9.9	3.53004E-07	3.43231E-07	2.58437E-07	2.03036E-07
10	3.97942E-07	4.20973E-07	3.43014E-07	2.90781E-07

Theorem-3: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, d \neq c, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N}, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\tau) > 0, \lambda \neq v$, Then the solution of

$$N(t) - N_0 \left({}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{k, \zeta, \tau}^{\xi, \eta}(t) \right) \right) = -c^v {}_0D_t^{-v} N(t) \tag{2.6}$$

is given by

$$N(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \times \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1}(-c^v t^v) \tag{2.7}$$

Proof: By the approach of Laplace Transform on (2.6)

$$\begin{aligned} N(s) &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\quad \times (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1)} (1 + c^v s^{-v})^{-1} \\ &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\quad \times (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1)} \sum_{\alpha=0}^{\infty} (-c^v s^{-v})^\alpha \\ &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \left(\frac{1}{n+1} \right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \\ &\quad \times \sum_{k=0}^{\infty} (-c^v)^\alpha (s)^{-(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1 + v\alpha)} \end{aligned}$$

By the approach of inverse Laplace Transform

$$\begin{aligned}
 N(t) &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \\
 &\quad \times \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} \sum_{\alpha=0}^{\infty} \frac{(-c^v t^v)^\alpha}{\Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1 + v\alpha)} \\
 &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r (\xi)_{m\eta, k} \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma(k(m\zeta + \tau))} \\
 &\quad \times \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1}(-c^v t^v)
 \end{aligned}$$

Graphical Numerical Interpretation of equation 2.7

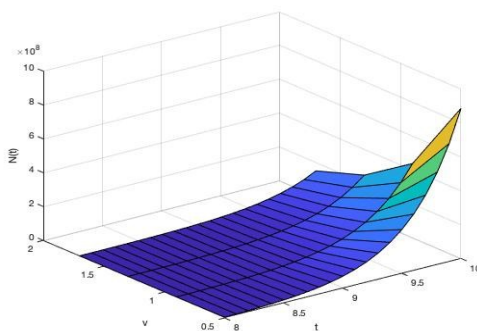


Figure -3(A)

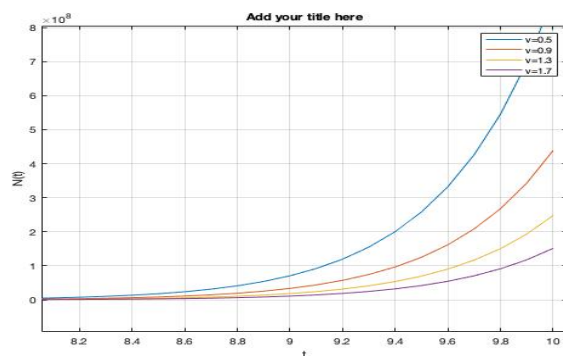


Figure -3(B)

Table 3: The values of $N(t)$ in GFKE in equation 2.7 at several values of t and v

t	Fix $v = 0.5$	Fix $v = 0.9$	Fix $v = 1.3$	Fix $v = 1.7$
	$N(t)$	$N(t)$	$N(t)$	$N(t)$
8	4173264.295	1891384.632	1000490.102	571592.1546
8.1	5622022.125	2559602.285	1358850.332	779234.498
8.2	7546399.737	3451330.031	1838793.491	1058346.918
8.3	10093814.6	4637244.916	2479337.516	1432205.793
8.4	13454742.9	6209117.935	3331326.19	1931249.601
8.5	17874547.06	8285745.433	4460808.627	2595157.609
8.6	23668351.46	11020448.72	5953327.673	3475496.621
8.7	31239701.21	14610530.92	7919346.806	4639081.649
8.8	41103904.37	19309170.22	10501099.49	6172231.527
8.9	53917157.84	25440338.8	13881211.32	8186143.488
9	70512798.13	33417469.73	18293526.07	10823663.38
9.1	91946308.58	43766755.73	24036665.28	14267792.35
9.2	119551062.9	57156159.01	31490970.2	18752348.89
9.3	155007203	74431446.45	41139619.3	24575300.14
9.4	200426547.7	96660848.01	53594889.22	32115391.08
9.5	258457027.2	125190276.3	69630737.52	41852839.7
9.6	332410845.6	161711453.4	90223138.55	54395033.49
9.7	426421421.1	208345778.8	116599908.5	70508365.66
9.8	545635153.4	267747355.6	150302119.7	91157591.87
9.9	696445255.8	343229286.7	193259641.5	117554380.8
10	886776291.7	438918178.9	247883865.6	151217081.4

Theorem-4: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, d \neq c, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N}, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\tau) > 0, \lambda \neq v, d \neq c$. Then the solution of

$$N(t) - N_0 \left({}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)} (c^v t^v) E_{k, \zeta, \tau}^{\xi, \eta} (c^v t^v) \right) \right) = -d^v {}_0D_t^{-v} N(t) \tag{2.8}$$

is given as

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \times \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times (t)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda} E_{v, v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}(-d^v t^v) \tag{2.9}$$

Proof: By the approach of Laplace Transform on both sides of (2.8), we have

$$\begin{aligned} N(s) &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ &\times \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times \frac{1}{s^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}} (1 + d^v s^{-v})^{-1} \\ &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ &\left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times \frac{1}{s^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}} \sum_{\alpha=0}^{\infty} (-d^v s^{-v})^\alpha \\ &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ &\left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times \frac{1}{s^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}} \sum_{\alpha=0}^{\infty} (-d^v s^{-v})^\alpha \end{aligned}$$

Now taking inverse Laplace Transform, we have

$$\begin{aligned} N(t) &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ &\times \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times (t)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda} \\ &\times \sum_{\alpha=0}^{\infty} \frac{(-d^v t^v)^\alpha}{\Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1 + v\alpha)} \\ &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ &\times \left(\frac{c}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times (t)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda} \\ &\times E_{v, v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}(-d^v t^v) \end{aligned}$$

Graphical Numerical Interpretation of equation 2.9

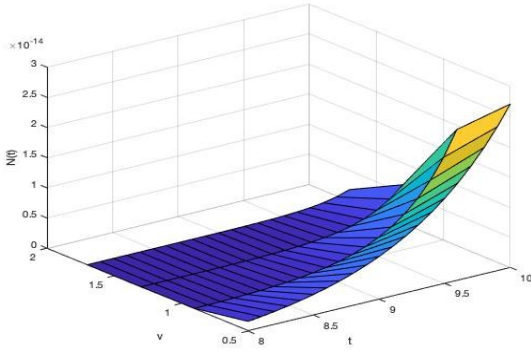


Figure-4(A)

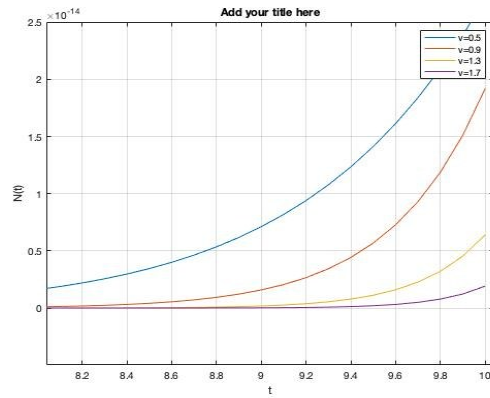


Figure-4(B)

Table 4: The values of $N(t)$ in GFKE in equation 2.9 at several values of t and v

t	Fix $v = 0.5$	Fix $v = 0.9$	Fix $v = 1.3$	Fix $v = 1.7$
	$N(t)$	$N(t)$	$N(t)$	$N(t)$
8	1.60253E-15	9.86577E-17	3.29704E-18	9.70781E-20
8.1	1.87461E-15	1.32059E-16	5.01643E-18	1.68176E-19
8.2	2.18884E-15	1.76172E-16	7.59464E-18	2.89416E-19
8.3	2.55113E-15	2.34249E-16	1.14423E-17	4.94843E-19
8.4	2.96814E-15	3.10472E-16	1.71581E-17	8.40749E-19
8.5	3.44735E-15	4.10206E-16	2.56104E-17	1.41966E-18
8.6	3.9972E-15	5.40319E-16	3.80548E-17	2.3828E-18
8.7	4.62713E-15	7.09573E-16	5.62978E-17	3.97588E-18
8.8	5.3477E-15	9.2912E-16	8.2929E-17	6.59603E-18
8.9	6.17077E-15	1.21312E-15	1.21646E-16	1.08816E-17
9	7.10955E-15	1.57949E-15	1.77709E-16	1.78533E-17
9.1	8.17882E-15	2.05089E-15	2.58569E-16	2.91349E-17
9.2	9.39503E-15	2.65585E-15	3.74751E-16	4.72964E-17
9.3	1.07765E-14	3.43024E-15	5.41056E-16	7.6386E-17
9.4	1.23437E-14	4.41906E-15	7.78238E-16	1.22749E-16
9.5	1.41192E-14	5.67864E-15	1.11529E-15	1.96285E-16
9.6	1.61283E-14	7.27928E-15	1.59259E-15	3.12369E-16
9.7	1.83987E-14	9.3086E-15	2.26618E-15	4.94769E-16
9.8	2.09614E-14	1.18756E-14	3.21357E-15	7.80071E-16
9.9	2.38507E-14	1.51153E-14	4.54168E-15	1.22435E-15
10	2.71042E-14	1.91952E-14	6.39752E-15	1.91317E-15

3. Special Cases

(i) The equation (2.4) reduces in value if we assume that $d=c$ in Theorem 2.

$$N(t) - N_0 \left\{ \mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^v t^v) E_{k, \zeta, \tau}^{\xi, \eta}(c^v t^v) \right\} = -c^v {}_0D_t^{-v} N(t) \tag{3.1}$$

and the solution to this is provided by

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \times \left(\frac{ct}{n+1} \right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times E_{v, (v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}(-d^v t^v) \tag{3.2}$$

(ii) The equation (2.4) reduces in value if we assume that $c = 1$ in Theorem 2,

$$N(t) - N_0 N_0 \left\{ \mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t^\nu) E_{k, \zeta, \tau}^{\xi, \eta}(t^\nu) \right\} = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.3}$$

and the solution to this is provided by

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ \times \left(\frac{t}{n+1}\right)^{\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times E_{\nu, (\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}(-d^\nu t^\nu) \tag{3.4}$$

(iii) The equation (2.8) reduces in value if we assume that $d = c$ in Theorem 4,

$$N(t) - N_0 \left({}_0D_t^\lambda \left(\mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^\nu t^\nu) E_{k, \zeta, \tau}^{\xi, \eta}(c^\nu t^\nu) \right) \right) = -c^\nu {}_0D_t^{-\nu} N(t) \tag{3.5}$$

and its solution is given by

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ \times \left(\frac{c}{n+1}\right)^{\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times (t)^{\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda} \\ \times E_{\nu, \nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}(-c^\nu t^\nu) \tag{3.6}$$

(iv) The equation (2.8) reduces in value if we assume that $d = c$ in Theorem 4,

$$N(t) - N_0 \left({}_0D_t^\lambda \left(\mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t^\nu) E_{k, \zeta, \tau}^{\xi, \eta}(t^\nu) \right) \right) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.7}$$

and its solution is given by

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, k}}{k! \Gamma(k(m\zeta + \tau))} \frac{(-1)^r \Gamma(\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \\ \times \left(\frac{1}{n+1}\right)^{\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \times (t)^{\nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda} \\ \times E_{\nu, \nu(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) - \lambda + 1}(-c^\nu t^\nu) \tag{3.8}$$

(v) When $\eta = 1, r = 1, \xi = 1$, then Mittag-Leffler function [28,31] represented as follow

$$E_{1, \zeta, \tau}^{1, 1}(t) = \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + \tau))} t^m \tag{3.9}$$

Further, in view of Theorem 1, 2, 3 and 4 we readily get following results

Corollary 1 If $c > 0, \nu > 0, |t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}$, then the solution of the equation

$$N(t) - N_0 \left(\mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{1, \zeta, \tau}^{1, 1}(t) \right) = -c^\nu {}_0D_t^{-\nu} N(t) \tag{3.10}$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + \tau))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{r(n+1) + m}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! (n+1)^{r(n+1)}} \\ \times E_{\nu, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}(-c^\nu t^\nu) \tag{3.11}$$

Corollary 2 If $c > 0, \nu > 0, |t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}$, then the solution of the equation

$$N(t) - N_0 \left(\mathfrak{S}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^\nu t^\nu) E_{1, \zeta, \tau}^{1, 1}(c^\nu t^\nu) \right) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.12}$$

$$\begin{aligned}
 &N(t) \\
 &= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + \tau))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{ct}{n+1}\right)^{v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1))} \\
 &\times E_{v, (v(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m) + 1)}(-d^v t^v)
 \end{aligned}
 \tag{3.13}$$

Corollary 3 If $c > 0, v > 0, |t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}$, then the solution of the equation

$$N(t) - N_0 {}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{1, \zeta, \tau}^{1,1}(t) \right) = -c^v {}_0D_t^{-v} N(t)$$

...(3.14)

is given by

$$\begin{aligned}
 N(t) = &\sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! \Gamma((m\zeta + \tau))} \\
 &\times \left(\frac{1}{n+1}\right)^{\tau_1 + \tau_2 + \dots + \tau_n + r(n+1)} E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m - \lambda + 1}(-c^v t^v)
 \end{aligned}$$

...(3.15)

Corollary 4 If $c > 0, v > 0, |t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}$, then the solution of the equation

$$N(t) - N_0 {}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^v t^v) E_{1, \zeta, \tau}^{1,1}(c^v t^v) \right) = -d^v {}_0D_t^{-v} N(t)$$

...(3.16)

$$\begin{aligned}
 N(t) = &N_0 \sum_{m=0}^{\infty} \frac{(\xi)_{m\eta, r}}{r! \Gamma(r(m\zeta + \tau))} \sum_{n=0}^{\infty} \frac{(-\beta)^n \Gamma((2n + p + 1)v + 1) t^{(2n+p+1)v + m - \lambda}}{\Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + p + \frac{(b+2)}{2}\right)} \left(\frac{c^v}{2}\right)^{2n+p+1} \\
 &\times E_{v, ((2n+p+1)v + m - \lambda + 1)}(-d^v t^v)
 \end{aligned}$$

...(3.17)

1. When $\eta = 1, r = 1, \xi = 1, \tau = 1$, then Mittag-Leffler function [28,29] represented as follow

$$E_{1, \zeta, 1}^{1,1}(t) = \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + 1))} t^m$$

...(3.18)

Furthermore, we may readily obtain the following findings in light of the aforementioned substitution and Theorems 1, 2, 3, and 4.

Corollary 5 If $|t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}, c > 0, v > 0$ then the solution of the equation

$$N(t) - N_0 \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{1, \zeta, 1}^{1,1}(t) \right) = -c^v {}_0D_t^{-v} N(t)$$

...(3.19)

is given by

$$\begin{aligned}
 &N(t) \\
 &= N_0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + 1))} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1) t^{r(n+1) + m}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! (n+1)^{r(n+1)}} \\
 &\times E_{v, \tau_1 + \tau_2 + \dots + \tau_n + r(n+1) + m + 1}(-c^v t^v)
 \end{aligned}$$

...(3.20)

Corollary 6 If $|t| < \infty, \Re e(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}, c > 0, v > 0$ then the solution of the equation

$$N(t) - N_0 \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^v t^v) E_{1, \zeta, 1}^{1,1}(c^v t^v) \right) = -d^v {}_0D_t^{-v} N(t)$$

...(3.21)

is given by

$$N(t)$$

$$= N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + 1))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n + 1) + m) + 1) t^m}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \times \left(\frac{ct}{n+1}\right)^{v(\tau_1+\tau_2+\dots+\tau_n+r(n+1))} \times E_{v,(v(\tau_1+\tau_2+\dots+\tau_n+r(n+1)+m)+1)}(-d^v t^v) \dots(3.22)$$

Corollary 7 If $|t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}, c > 0, v > 0$ then the solution of the equation

$$N(t) - N_0 {}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(t) E_{1, \zeta, 1}^{1, 1}(t) \right) = -c^v {}_0D_t^{-v} N(t) \dots(3.23)$$

is given by

$$N(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^r \Gamma(\tau_1 + \tau_2 + \dots + \tau_n + r(n + 1) + m + 1) t^{\tau_1+\tau_2+\dots+\tau_n+r(n+1)+m-\lambda}}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r! k! \Gamma((m\zeta + 1))} \times \left(\frac{1}{n+1}\right)^{\tau_1+\tau_2+\dots+\tau_n+r(n+1)} E_{v, \tau_1+\tau_2+\dots+\tau_n+r(n+1)+m-\lambda+1}(-c^v t^v) \dots(3.24)$$

Corollary 8 If $|t| < \infty, \Re(\tau_i + 1) > 0, t, \tau_i, \zeta \in \mathbb{C}, c > 0, v > 0$ then the solution of the equation

$$N(t) - N_0 {}_0D_t^\lambda \left(\mathfrak{J}_{\tau_1, \tau_2, \dots, \tau_n}^{(n)}(c^v t^v) E_{1, \zeta, 1}^{1, 1}(c^v t^v) \right) = -d^v {}_0D_t^{-v} N(t) \dots(3.25)$$

is given by

$$N(t) = N_0 \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\Gamma((m\zeta + 1))} \frac{(-1)^r \Gamma(v(\tau_1 + \tau_2 + \dots + \tau_n + r(n + 1) + m) + 1)}{\Gamma(r + \tau_1 + 1) \dots \Gamma(r + \tau_n + 1) r!} \left(\frac{c}{n+1}\right)^{v(\tau_1+\tau_2+\dots+\tau_n+r(n+1))} \times (t)^{v(\tau_1+\tau_2+\dots+\tau_n+r(n+1)+m)-\lambda} E_{v, v(\tau_1+\tau_2+\dots+\tau_n+r(n+1)+m)-\lambda+1}(-d^v t^v) \dots(3.26)$$

Although we don't explicitly record them here, further specific examples of our results can be achieved by entering appropriate parametric variables.

Conclusion

The propose of this paper that acquire the solution of generalized fractional kinetic equations associated with Hper-Bessel Function and K-Mittag-Leffler Function by using Integral Transform. We have discussed aspect of the four results (1-4) which have several applications in the field of Applied Mathematics, Applied Physics and Engineering. Further point out the graphical interpretation of results on unique parameter values at certain time periods.

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