UtilitasMathematica

ISSN 0315-3681 Volume 122 (2), 2025

A NOVEL APPROACH TO FRACTIONAL KINETIC EQUATIONS INVOLVING LAGUERRE POLYNOMIAL FUNCTION AND S-FUNCTION

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Abstract

Fractional kinetic equations incorporating special functions have proven useful in explaining and solving many significant mathematical and mathematical physics problems. Given the significant role of arbitrary-order kinetic equations, this study focuses on solving a newly formulated equation of this type by utilizing the Sumudu transform. The equation incorporates fractional derivatives and involves a composition of Laguerre polynomials and the S-Function. Our investigation included MATLAB-generated graphical representations to show how these solutions behave under different parametric conditions. It is important to highlight that the study's results are incredibly flexible and could lead to confirmed and perhaps undiscovered research findings in this area.

Keywords: Generalized Fractional Kinetic Equation, Sumudu Transform, Laguerre Polynomials, S-Function, Mittag-Leffler Function.

2010 Mathematics Subject Classification: 26A33, 33E12, 33E20, 44A99

1. Introduction

Fractional calculus, which involves fractional derivatives and integrals, provides the mathematical framework for addressing fractional differential equations. Recently, fractional differential equations have been utilized to create mathematical models of a wide range of physical processes and are increasingly significant in the applied sciences, particularly in dynamical systems, control systems, mathematical physics, and engineering [1,2]. However, solving fractional differential equations is often complex due to the nonlocal nature of fractional derivatives. As a result, numerical techniques, such as the Mittag-Leffler function, are commonly employed to approach these findings. Researchers are continuously developing new techniques and tools to improve the understanding and solutions of these equations due to their importance in the modelling of complex physical and biological systems [1,3,4]

Many researchers have worked on defining fractional derivatives, with most approaches based on integral formulations such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz, and Weyl derivatives. Since these definitions are derived from fractional integrals, they inherently exhibit nonlocal properties. Heredity and non-locality are key characteristics of these definitions, making them essential for many applications and distinguishing them from the classical Newton-Leibniz calculus.

In particular, kinetic equations serve as core components in fundamental equations of mathematical physics and the natural sciences since they specify the continuity of motion of matter [5-13]. Based on the idea of fractional calculus, FKEs are a class of differential equations that explain how a system changes over time. Modelling anomalous diffusion is one of the most important uses of FKEs. A form of diffusion known as anomalous diffusion occurs when a particle's mean-square displacement increases sublinearly over time. Numerous systems, such as porous media, turbulent fluids, and biological tissues, exhibit this kind of diffusion. Additionally, a wide range of other systems have been modeled using FKEs, including the behavior of financial markets, the evolution of traffic patterns, the transmission of illness, and the relaxation of viscoelastic materials. Haubold and Mathai [14] introduced a fractional differential equation that links reaction rates with production and destruction rates.

$$\frac{dF}{dt} = -d(F_t) + p(F_t) \tag{1.1}$$

where $F = F(t)$ represents the reaction rate, $d = d(F)$ denotes the destruction rate, $p = p(F)$ indicates the production rate and $F_i t$ is defined by the function $F_t(t^*) = F_t(t - t^*), t^* > 0$. Furthermore, disregarding the inhomogeneity in quantity $F(t)$, equation (1.1) can be inserted.

$$\frac{dF_i}{dt} = -\xi_\nu F_i(t) \tag{1.2}$$

where the initial condition $F_i = F_0$ represents the density number of index i at time $t = 0$ and $c_\nu = 0$,then the solution of (1.2) can be referred to.

$$F(t) = F_0 e^{\xi_i t} \tag{1.3}$$

Furthermore, [15] presented the FKE in the following form

$$F(t) - F_0 = \xi^{\omega}\, {}_0D_t^{-\omega} F(t) \tag{1.4}$$

Here, $F(t)$ represents the density of species at time t, $F(0) = F_0$ denoting the density at time $t = 0$, and the parameter ξ is a constant, where $f(t)$ belongs to space $L(0, \infty)$

And ${}_0D_t^{-\omega}$ refers to the fractional integral operator, mathematically defined as

$${}_0D_t^{-\omega} f(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t - \varphi)^{\omega - 1} f(\varphi) d\varphi, t > 0, \Re(\omega) > 0 \tag{1.5}$$

In recent decades, Laguerre polynomials have attracted growing interest from engineers and scientists owing to their extensive use in solving a variety of practical and applied problems. For a comprehensive discussion on their properties, generalizations and applications, both in the classical sense and within the framework of fractional calculus, readers may refer to earlier works [16-18]. Furthermore, these polynomials have been applied in solving various problems in mathematical physics, including delay differential equations [19], Volterra pantograph-type integrodifferential equations, and fractional differential equations.

The Fractional Laguerre polynomials $L_r^f(t)$ are the generalized classical Laguerre polynomials [20], which are solutions to Laguerre's differential equations and are used in many physical problems. The fractional Laguerre polynomial (FLP), the fractional form of the Laguerre polynomial (LP), is also very important and useful in electromagnetic wave propagation via transmission lines [21-23]. The generating functions, orthogonality features, and recurrence relations of fractional Laguerre polynomials (FLPS) are comparable to those of classical Laguerre polynomials.

Definition 1.1 For $r \in \nu$ and $r - 1 < \omega < r, 0 \leq t < \infty \; and \; f > -1$, the Fractional Laguerre Polynomials [20] are defined by the following expression:

$$L_r^f(t) = \sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}; 0 \leq t < \infty \qquad (1.6)$$

equation (1.6) can be further expressed in terms of hypergeometric functions as

$$L_r^f(t) = \sum_{c=0}^{\mathrm{r}} \frac{\Gamma(r+f+1)}{\Gamma(f+1)\Gamma(r+1)}\, {}_1F_1(-r;(f+1);t) \qquad (1.7)$$

here the function ${}_1F_1(p;q;z)$ also known as Kummer's function [24], is the confluent hypergeometric function of first kind, defined as

$${}_1F_1(p;q;z) = \sum_{c=0}^{\infty} \frac{(\mathrm{p})_\alpha}{(\mathrm{q})_\alpha} \frac{z^\alpha}{\alpha!} \qquad (1.8)$$

where $(r)_\alpha$ is the Pochhamer symbol and is given as follows

$$r_f = \frac{\Gamma(\mathrm{r}+\alpha)}{\Gamma(\mathrm{r})} \qquad (1.9)$$

Remark: The fractional Laguerre polynomials [20] defined in equation (1.7) reduce to simple Laguerre polynomials, for $f = 0$, which is expressed as

$$L_r(t) = \sum_{c=0}^{\mathrm{r}} \frac{\Gamma(r+1)t^c}{[\Gamma(c+1)]^2\Gamma(r-c+1)}; r \in \mathrm{v}, 0 \leq \mathrm{t} < \infty \qquad (1.10)$$

The well-known result from equation (1.1) is as follows:

$${}_0D_t^{-\gamma} t^c = \frac{\Gamma(\mathrm{c}+1)}{\Gamma(\mathrm{c}+\gamma+1)} t^{c+\gamma}, \Re(\mathrm{c}) > -1, 0 < \Re(\gamma) < 1, \mathrm{t} > 0 \qquad (1.11)$$

Thus, considering equations (1.6) and (1.11), we have

$${}_0D_t^{-\gamma}\left[L_r^f(t)\right] = \sum_{c=0}^{\infty} \frac{(-1)^{\mathrm{c}}(1+f)_r t^{c+\gamma}}{(\mathrm{f}+1)_{\mathrm{c}}\Gamma(r-c+1)\Gamma(c+\gamma+1)} \qquad (1.12)$$

Further,

$${}_0D_t^{-\gamma}\left[L_r^f(\xi^\omega t^\omega)\right] = \sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}(1+f)_r \xi^{\omega c}\Gamma(\xi c+1)t^{\omega c+\gamma}}{(\mathrm{f}+1)_{\mathrm{c}}\Gamma(c+1)\Gamma(r-c+1)\Gamma(\omega c+\gamma+1)} \qquad (1.13)$$

In recent developments, Saxena and Daiya [25] introduced the S-function, a newly defined special function, and examined its connections with several existing functions in mathematical analysis. This function acts as a unifying framework, encompassing the k-Mittag-Leffler function, K-function, M-series, and the classical Mittag-Leffler function as particular cases. Due to their versatility and effectiveness, such generalized functions have seen a surge in relevance for modelling and problem-solving in diverse areas, including applied sciences, biological systems, physics, and engineering.

In addition, the S-function [26], originally introduced by G. S. Sánchez in 2010, has demonstrated effectiveness in addressing fractional kinetic equations, particularly because of its capability to represent non-local and non-linear dynamics [15,27], Expanding and generalizing fractional kinetic equations (FKEs) through the use of fractional operators has opened new avenues for solving fractional differential equations, especially those involving the Riemann-Liouville fractional derivative.

Definition 1.2 The S-function [25] is characterized by the expression provided below

$$\mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_{\mathrm{v}};t) = \sum_{\mathrm{v}=0}^{\infty} \frac{(\mathrm{s}_1)_{\mathrm{v}}(s_2)_{\mathrm{v}}\ldots(s_u)_{\mathrm{v}}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_{\mathrm{v}}(\mathrm{r}_2)_{\mathrm{v}}\ldots(\mathrm{r}_{\mathrm{v}})_{\mathrm{v}}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)} \frac{(t)^{\mathrm{v}}}{\mathrm{v}!} \qquad (1.14)$$

here $\alpha, \theta, \rho, \eta, \kappa \in \mathbb{C}, \Re(\theta) > 0, \mathrm{k} \in \Re, \Re(\theta) > \mathrm{k}\,\Re(\alpha), \mathrm{s_i}(i = 1,2,3,\ldots.,u), t_j(j = 1,2,3,\ldots,v)$ and $u < v + 1$

Below, we present several particular instances of the S function are outlined as

(i) For $u = 0$ and $\omega = 0$, the function reduced in the generalized k-Mittag-Leffler function [28] as

$$\mathrm{E}_{\kappa,\theta,\eta}^{\rho,\alpha}(t) = \mathrm{S}_{(0,0)}^{(\theta,\eta,\rho,\alpha,\kappa)}(-;-;t) = \sum_{\mathrm{v}=0}^{\infty} \frac{(\rho)_{\mathrm{v}\alpha,\kappa}}{\Gamma_{\kappa}(\mathrm{v}\theta+\eta)} \frac{(t)^{\mathrm{v}}}{\mathrm{v}!}, \Re\left(\frac{\rho}{\kappa}-\alpha\right) > (u-v)$$

(ii) For $\kappa = \alpha = 1$, the function reduced in the K- function [29] as

$$\begin{aligned} \mathrm{K}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho)}(s_1,s_2,..s_u;r_1,r_2,..r_{\mathrm{v}};t) &= \mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,1,1)}(s_1,s_2,..s_u;r_1,r_2,..r_{\mathrm{v}};t) \\ &= \sum_{\mathrm{v}=0}^{\infty} \frac{(\mathrm{s}_1)_{\mathrm{v}}(s_2)_{\mathrm{v}} \ldots (s_u)_{\mathrm{v}}(\rho)_{\mathrm{v}}}{(\mathrm{r}_1)_{\mathrm{v}}(\mathrm{r}_2)_{\mathrm{v}} \ldots (\mathrm{r}_{\mathrm{v}})_{\mathrm{v}}\, \Gamma_{\kappa}(\mathrm{v}\theta+\eta)} \frac{(t)^{\mathrm{v}}}{\mathrm{v}!}, \Re(\theta) = (\mathrm{u}-\mathrm{v}) \end{aligned}$$

(iii) For $\alpha = \kappa = \rho = 1$, the function can be expressed as M-series [30]

$$\begin{aligned} \mathrm{M}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta)}(s_1,s_2,..s_u;r_1,r_2,..r_{\mathrm{v}};t) &= \mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,1,1,1)}(s_1,s_2,..s_u;r_1,r_2,..r_{\mathrm{v}};t) \\ &= \sum_{\mathrm{v}=0}^{\infty} \frac{(\mathrm{s}_1)_{\mathrm{v}}(s_2)_{\mathrm{v}} \ldots (s_u)_{\mathrm{v}}}{(\mathrm{r}_1)_{\mathrm{v}}(\mathrm{r}_2)_{\mathrm{v}} \ldots (\mathrm{r}_{\mathrm{v}})_{\mathrm{v}}\, \Gamma_{\kappa}(\mathrm{v}\theta+\eta)} \frac{(t)^{\mathrm{v}}}{\mathrm{v}!}, \Re(\theta) = \mathrm{u}-\mathrm{v}-1 \end{aligned}$$

(iv) For $\alpha = \kappa = \rho = 1$, and $u = v = 0$ the function reduces in the generalized Mittag-Leffler function [28] as

$$\mathrm{E}_{\theta,,\eta} = \mathrm{S}_{(0,0)}^{(\theta,\eta,1,1,1)}(-;-;t) = \sum_{\mathrm{v}=0}^{\infty} \frac{1}{\Gamma(\mathrm{v}\theta+\eta)} \frac{(t)^{\mathrm{v}}}{\mathrm{v}!}, \Re(\theta) > 0, \Re(\eta) > 0$$

In this study, we utilize the Sumudu transform to analyze fractional integrals and derivatives, in order to solve fractional differential equations with initial values. Many researchers have explored various properties of the Sumudu transform, leading to the development of efficient and straightforward methodologies for addressing ordinary and partial differential equations. The growing interest in this transformation highlights its potential applications in diverse mathematical and physical sciences problems. Due to its linear and bilateral nature, along with its scale and unit-preserving properties, the Sumudu transform proves to be a powerful tool for solving various difference and differential equations without requiring a new frequency domain.

Definition 1.3 [31] The Sumudu transform is a powerful integral transform technique that is defined over the set of functions

$$R = \{f(t) | \exists \zeta, v_1, v_2 > 0, |f(t)| < \zeta e^{\frac{|t|}{\varphi}}, t \in (-1)^c \times [0,\infty)\}$$

which is defined as follows

$$G(\varphi) = S[f(t);\varphi] = \int_0^{\infty} e^{-t} f(\varphi t)\, dt; \varphi \in (v_1, v_2) \tag{1.15}$$

The following definition provides the Sumudu transform corresponding to the Riemann–Liouville fractional integral operator as follows

$$S\{{}_0D_t^{-\omega} f(t);\varphi\} = \varphi^{\omega} G(\varphi) \tag{1.16}$$

Our results demonstrate the benefits of applying the Sumudu transform over conventional techniques, especially in terms of its smooth handling of the fractional derivative operator and its compatibility with a variety of functions, including special functions like the S-function and Laguerre polynomials. This approach simplifies the computational process while providing a deeper understanding of the behavior of solutions to fractional kinetic equations. An important advancement in the study of fractional kinetic equations with the Sumudu transform provides a strong foundation for applied mathematics and engineering scholars and practitioners.

2. Main Results

In this technique, some new FKEs are proposed, and their solutions are acquired by the Sumudu Transform. Moreover, the numerical and graphical interpretation of these findings is presented in the subsequent part.

Theorem 1: If $\omega > 0, \xi > 0$ *and* $r \in \nu, r - 1, 0 \leq t < \infty$ *and* $f > -1$, with the conditions of (1.13), then the FKE

$$F(t) - F_0\left\{L_r^f(t)\mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_\mathrm{v}; t)\right\} = -\xi^{\omega}{}_0D_t^{-\omega}F(t) \qquad (2.1)$$

yielding the solution

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{v}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{v}(s_2)_\mathrm{v}\dots(s_u)_\mathrm{v}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_\mathrm{v}(\mathrm{r}_2)_\mathrm{v}\dots(\mathrm{r}_\mathrm{v})_\mathrm{v}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)}\frac{(t)^\mathrm{v}}{\mathrm{v}!}\right)$$

$$\times\,\Gamma(\mathrm{c}+\mathrm{v}+1)\times\frac{1}{\mathrm{t}}E_{\omega,c+\mathrm{v}}(-\xi^{\omega}t^{\omega}) \qquad (2.2)$$

Proof: By employing the Sumudu Transform technique on (2.1) and incorporating (1.15), we obtain

$$S(F(t);\varphi) = F_0S\left\{L_r^f(t)\mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_\mathrm{v}; t);\varphi\right\} = -\xi^{\omega}S({}_0D_t^{-\omega}F(t);\varphi) \qquad (2.3)$$

which is further given as

$$\bar{F}(\varphi)$$
$$= F_0\int_0^{\infty}e^{-t}\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\varphi t)^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{v}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{v}(s_2)_\mathrm{v}\dots(s_u)_\mathrm{v}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_\mathrm{v}(\mathrm{r}_2)_\mathrm{v}\dots(\mathrm{r}_\mathrm{v})_\mathrm{v}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)}\frac{(\varphi t)^\mathrm{v}}{\mathrm{v}!}\right)dt$$
$$-\xi^{\omega}(\varphi)^{\omega}F(\varphi) \qquad (2.4)$$

Solved (2.4) by reordering the integration and summation, the expression is simplified as follows:

$$\bar{F}(\varphi)[1+\xi^{\omega}(\varphi)^{\omega}]$$
$$= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\varphi^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{v}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{v}(s_2)_\mathrm{v}\dots(s_u)_\mathrm{v}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_\mathrm{v}(\mathrm{r}_2)_\mathrm{v}\dots(\mathrm{r}_\mathrm{v})_\mathrm{v}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)}\frac{(\varphi)^\mathrm{v}}{\mathrm{v}!}\right)$$
$$\times\int_0^{\infty}e^{-t}\,t^{c+\mathrm{v}}dt$$

$$= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\varphi^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{v}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{v}(s_2)_\mathrm{v}\dots(s_u)_\mathrm{v}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_\mathrm{v}(\mathrm{r}_2)_\mathrm{v}\dots(\mathrm{r}_\mathrm{v})_\mathrm{v}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)}\frac{(\varphi)^\mathrm{v}}{\mathrm{v}!}\right)$$

$$\times\,\Gamma(\mathrm{c}+\mathrm{v}+1) \qquad (2.5)$$

which leads to

$$\bar{F}(\varphi) = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)1^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{v}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{v}(s_2)_\mathrm{v}\dots(s_u)_\mathrm{v}(\rho)_{\mathrm{v}\alpha,\kappa}}{(\mathrm{r}_1)_\mathrm{v}(\mathrm{r}_2)_\mathrm{v}\dots(\mathrm{r}_\mathrm{v})_\mathrm{v}\,\Gamma_\kappa(\mathrm{v}\theta+\eta)}\frac{(1)^\mathrm{v}}{\mathrm{v}!}\right)$$

$$\times \Gamma(\mathrm{c}+\nu+1) \;\; \times (\varphi)^{\mathrm{c}+\nu} \sum_{\mu=0}^{\infty}(-1)^{\mu}\,(\xi\varphi)^{\mu\omega} \tag{2.6}$$

Now, apply the Inverse Sumudu transform technique in (2.6) and use the property:

$$S^{-1}\{\varphi^{\omega};t\}=\frac{t^{\omega-1}}{\Gamma(\omega)};\Re(\omega)>0 \tag{2.7}$$

We obtain

$$F(t)=F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)1^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(1)^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\mathrm{c}+\nu+1)\;\;\times\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}\,(\xi)^{\mu\omega}\,\frac{t^{\mu\omega+c+\nu-1}}{\Gamma(\mu\omega+c+\nu)}\right)$$

$$=F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(t)^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\mathrm{c}+\nu+1)\;\;\times\frac{1}{\mathrm{t}}\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{(\xi t)^{\mu\omega}}{\Gamma(\mu\omega+c+\nu)}\right) \tag{2.9}$$

Now, Equation (2.9) can be written as follows:

$$F(t)=F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(t)^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\mathrm{c}+\nu+1)\times\frac{1}{\mathrm{t}}E_{\omega,c+\nu}(-\xi^{\omega}t^{\omega})$$

By using the specific values of parameters t and ω, the numerical analysis of the results can be studied through Table 1 and Figure 1 (i) and Figure 1 (ii) to obtain the graphical representation of the finding and behavior.

The values of $F(t)$ corresponding to various values of t and ω for equation (2.2)

t	**F(t) at $\omega = 0.1$**	**F(t) at $\omega = 0.5$**	**F(t) at $\omega = 0.9$**	**F(t) at $\omega = 1.3$**
4.0	0.00201	−0.00028	−0.013951	−0.025662
4.2	0.003152	0.003187	−0.013499	−0.030112
4.4	0.004618	0.008406	−0.01124	−0.034176
4.6	0.00647	0.015867	−0.006345	−0.03721
4.8	0.00878	0.02615	0.00227	−0.038288
5.0	0.01162	0.03995	0.016022	−0.036096
5.2	0.01507	0.058074	0.036628	−0.029829
5.4	0.01919	0.08141	0.06623	−0.01409
5.6	0.0241	0.11103	0.10747	0.011319
5.8	0.029883	0.1481303	0.16355	0.0518
6.0	0.036637	0.194045	0.23825	0.112077
6.2	0.04447	0.2503	0.33621	0.19964
6.4	0.05352	0.3186068	0.462487	0.322925
6.6	0.06389	0.400856	0.624564	0.49273
6.8	0.075728	0.499164	0.828868	0.722494
7.0	0.089163	0.618879	1.084565	1.028703
7.2	0.104347	0.753553	1.401619	1.431554
7.4	0.121436	0.915053	1.79191	1.95557
7.6	0.1405957	1.1034866	2.2688299	2.603631
7.8	0.161996	1.322258	2.847765	3.4914762

8.0	0.18582069	1.5750858	3.54626041	4.5813619

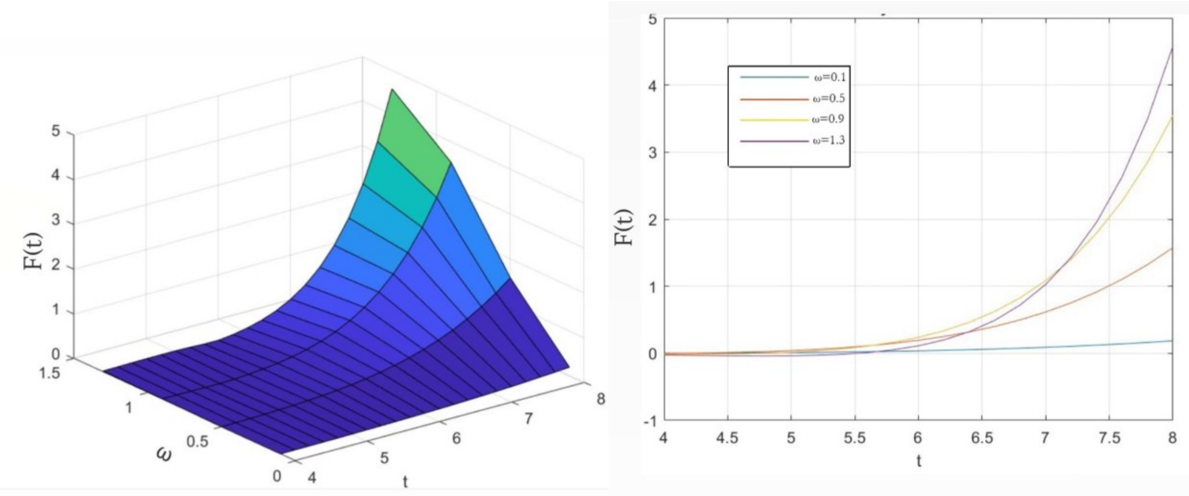

Figure 1 **(a) 3D graph for (2.2)** **(b) 2D graph for (2.2)**

Theorem 2: If $\omega > 0, \xi > 0, \xi \neq \varepsilon \; and \; r \in \nu, r-1, 0 \leq t < \infty \; and \; f > -1$, with the conditions of (1.13), then the FKE

$$F(t) - F_0\left\{L_r^f(\xi^\omega t^\omega)\mathrm{S}_{(\mathrm{u,v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_\mathrm{v};\xi^\omega t^\omega)\right\} = -\varepsilon^\omega{}_0D_t^{-\omega}F(t) \tag{2.11}$$

yielding the solution

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\xi t)^{\omega c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{\nu}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{\nu}(s_2)_\mathrm{\nu}\ldots(s_u)_\mathrm{\nu}(\rho)_{\mathrm{\nu\alpha},\kappa}}{(\mathrm{r}_1)_\mathrm{\nu}(\mathrm{r}_2)_\mathrm{\nu}\ldots(\mathrm{r_v})_\mathrm{\nu}\,\Gamma_\kappa(\mathrm{\nu\theta}+\eta)}\,\frac{(\xi t)^{\omega\mathrm{\nu}}}{\mathrm{\nu}!}\right)$$

$$\times\,\Gamma(\omega(\mathrm{c}+\mathrm{\nu})+1)\times\frac{1}{\mathrm{t}}E_{\omega,\omega c+\omega\mathrm{\nu}}(-\varepsilon^\omega t^\omega) \tag{2.12}$$

Proof: By employing the Sumudu Transform technique on (2.11) and incorporating (1.15), we obtain

$$S(F(t);\varphi) = F_0S\left\{L_r^f(\xi^\omega t^\omega)\mathrm{S}_{(\mathrm{u,v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_\mathrm{v};\xi^\omega t^\omega);\varphi\right\} = -\varepsilon^\omega S({}_0D_t^{-\omega}F(t);\varphi) \tag{2.13}$$

which is further given as

$$\bar{F}(\varphi) = F_0\int_0^\infty e^{-t}\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\left(\xi^\omega(\varphi t)^\omega\right)^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{\nu}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{\nu}(s_2)_\mathrm{\nu}\ldots(s_u)_\mathrm{\nu}(\rho)_{\mathrm{\nu\alpha},\kappa}}{(\mathrm{r}_1)_\mathrm{\nu}(\mathrm{r}_2)_\mathrm{\nu}\ldots(\mathrm{r_v})_\mathrm{\nu}\,\Gamma_\kappa(\mathrm{\nu\theta}+\eta)}\,\frac{\left(\xi^\omega(\varphi t)^\omega\right)^\mathrm{\nu}}{\mathrm{\nu}!}\right)dt$$
$$-\varepsilon^\omega(\varphi)^\omega F(\varphi) \tag{2.14}$$

Solved (2.14) by reordering the integration and summation, the expression is simplified as follows:

$$\bar{F}(\varphi)[1+\varepsilon^\omega(\varphi)^\omega] = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\;(\xi^\omega\varphi^\omega)^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\mathrm{\nu}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{\nu}(s_2)_\mathrm{\nu}\ldots(s_u)_\mathrm{\nu}(\rho)_{\mathrm{\nu\alpha},\kappa}}{(\mathrm{r}_1)_\mathrm{\nu}(\mathrm{r}_2)_\mathrm{\nu}\ldots(\mathrm{r_v})_\mathrm{\nu}\,\Gamma_\kappa(\mathrm{\nu\theta}+\eta)}\,\frac{(\xi^\omega\varphi^\omega)^\mathrm{\nu}}{\mathrm{\nu}!}\right)$$

$$\times \int_0^\infty e^{-t}\, t^{\omega(c+\mathrm{\nu})} dt$$

$$= F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\xi^\omega \varphi^\omega)^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)} \sum_{\nu=0}^{\infty} \frac{(\mathrm{s}_1)_\nu (s_2)_\nu \dots (s_u)_\nu (\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu (\mathrm{r}_2)_\nu \dots (\mathrm{r}_\mathrm{v})_\nu\, \Gamma_\kappa(\nu\theta+\eta)}\, \frac{(\xi^\omega \varphi^\omega)^\nu}{\nu!}\right)$$

$$\times \Gamma(\omega(\mathrm{c}+\nu)+1) \qquad (2.15)$$

which leads to

$$\bar{F}(\varphi) = F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\xi^\omega \varphi^\omega)^c}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)} \sum_{\nu=0}^{\infty} \frac{(\mathrm{s}_1)_\nu (s_2)_\nu \dots (s_u)_\nu (\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu (\mathrm{r}_2)_\nu \dots (\mathrm{r}_\mathrm{v})_\nu\, \Gamma_\kappa(\nu\theta+\eta)}\, \frac{(\xi^\omega \varphi^\omega)^\nu}{\nu!}\right)$$

$$\times \Gamma(\omega(\mathrm{c}+\nu)+1) \;\; \times \textstyle\sum_{\mu=0}^{\infty}(-1)^\mu\, (\varepsilon\varphi)^{\mu\omega} \qquad (2.16)$$

Now, apply the Inverse Sumudu transform technique in (2.16) and use (2.7):

We obtain

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\xi^{\omega c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)} \sum_{\nu=0}^{\infty} \frac{(\mathrm{s}_1)_\nu (s_2)_\nu \dots (s_u)_\nu (\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu (\mathrm{r}_2)_\nu \dots (\mathrm{r}_\mathrm{v})_\nu\, \Gamma_\kappa(\nu\theta+\eta)}\, \frac{\xi^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(\mathrm{c}+\nu)+1) \;\; \times \left(\textstyle\sum_{\mu=0}^{\infty}(-1)^\mu\, (\varepsilon)^{\mu\omega}\, \frac{t^{\omega(\mu+c+\nu)-1}}{\Gamma(\omega(\mu+c+\nu))}\right) \qquad (2.17)$$

$$= F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\xi t)^{\omega c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)} \sum_{\nu=0}^{\infty} \frac{(\mathrm{s}_1)_\nu (s_2)_\nu \dots (s_u)_\nu (\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu (\mathrm{r}_2)_\nu \dots (\mathrm{r}_\mathrm{v})_\nu\, \Gamma_\kappa(\nu\theta+\eta)}\, \frac{(\xi t)^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(\mathrm{c}+\nu)+1) \;\; \times \frac{1}{\mathrm{t}}\left(\textstyle\sum_{\mu=0}^{\infty}(-1)^\mu\, \frac{(\varepsilon t)^{\mu\omega}}{\Gamma(\omega(\mu+c+\nu))}\right) \qquad (2.18)$$

Now, Equation (2.18) can be written as follows:

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\xi t)^{\omega c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)} \sum_{\nu=0}^{\infty} \frac{(\mathrm{s}_1)_\nu (s_2)_\nu \dots (s_u)_\nu (\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu (\mathrm{r}_2)_\nu \dots (\mathrm{r}_\mathrm{v})_\nu\, \Gamma_\kappa(\nu\theta+\eta)}\, \frac{(\xi t)^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(\mathrm{c}+\nu)+1) \times \frac{1}{\mathrm{t}} E_{\omega,\omega c+\omega\nu}(-\varepsilon^\omega t^\omega) \qquad (2.19)$$

By using the specific values of parameters t and ω, the numerical analysis of the results can be studied through Table 2 and Figure 2 (i) and Figure 2 (ii) to obtain the graphical representation of the finding and behavior.

The values of $F(t)$ corresponding to various values of t and ω for equation (2.12)

t	**F(t) at $\omega = 0.1$**	**F(t) at $\omega = 0.5$**	**F(t) at $\omega = 0.9$**	**F(t) at $\omega = 1.3$**
4.0	$3.143e-05$	0.001233	−0.00325	−0.098316
4.2	$3.3025e-05$	0.00159	−0.002351	−0.12776
4.4	$3.4614e-05$	0.002019	−0.000637	−0.16246
4.6	$3.6197e-05$	0.00252	0.0022	−0.202031
4.8	$3.7773e-05$	0.003105	0.00655	−0.245346
5.0	$3.9344e-05$	0.003782	0.0129	−0.290201
5.2	$4.09095e-05$	0.00456	0.02184	−0.33286
5.4	$4.24688e-05$	0.00544	0.034089	−0.367484
5.6	$4.4022e-05$	0.00645	0.05048	−0.385374
5.8	$4.5571e-05$	0.00759	0.072052	−0.374047
6.0	$4.7114e-05$	0.00887	0.09999	−0.316055
6.2	$4.8653e-05$	0.0103	0.13573	−0.187531
6.4	$5.0186e-05$	0.01189	0.18091	0.04360643
6.6	$5.17157e-05$	0.01366	0.2374	0.41984
6.8	$5.32398e-05$	0.01562	0.30756	0.9967
7.0	$5.47595e-05$	0.01778	0.39378	1.845961
7.2	$5.62746e-05$	0.02015	0.49903	3.059454
7.4	$5.7785e-05$	0.02275	0.62661	4.753614
7.6	$5.92921e-05$	0.02559	0.780289	7.0748474
7.8	$6.07945e-05$	0.028698	0.96432	10.20585
8.0	$6.2293e-05$	0.0320693	1.183496	14.37306

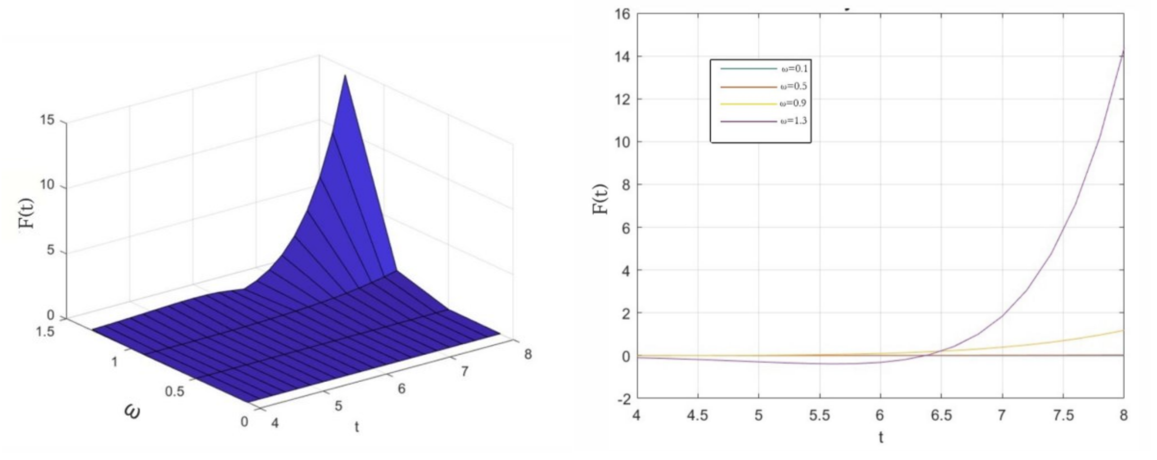

Figure 2 **(a) 3D graph for (2.12)** **(b) 2D graph for (2.12)**

Theorem 3: If $\omega > 0, \xi > 0, \gamma > 0$ and $r \in \nu, r-1 < \omega < r, 0 \leq t < \infty$ and $f > -1$, with the conditions of (1.13), then the FKE

$$F(t) - F_0\left[{}_0D_t^{-\gamma}\left\{L_r^f(t)\mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, ..s_u; r_1, r_2, ..r_\mathrm{v}; t)\right\}\right] = -\xi^\omega{}_0D_t^{-\omega}F(t) \tag{2.20}$$

yielding the solution

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^{c+\gamma}}{\Gamma(c+f+1)\Gamma(r-c+1)}\sum_{\mathrm{\nu}=0}^{\infty}\frac{(\mathrm{s}_1)_\mathrm{\nu}(s_2)_\mathrm{\nu}\dots(s_u)_\mathrm{\nu}(\rho)_{\mathrm{\nu\alpha},\kappa}}{(\mathrm{r}_1)_\mathrm{\nu}(\mathrm{r}_2)_\mathrm{\nu}\dots(\mathrm{r}_\mathrm{v})_\mathrm{\nu}\,\Gamma_\kappa(\mathrm{\nu\theta}+\eta)}\frac{(t)^\mathrm{\nu}}{\mathrm{\nu}!}\right)$$

$$\times (1+\mathrm{c})_{\nu} \times \frac{1}{\mathrm{t}} E_{\omega,c+\nu+\gamma}(\xi t)^{\omega} \tag{2.21}$$

Proof: By employing the Sumudu Transform technique on (2.20) and incorporating (1.15), we obtain

$$S(F(t);\varphi) = F_0 S\left[{}_0D_t^{-\gamma}\left\{L_r^f(t)\mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_\nu;t);\varphi\right\}\right] = -\xi^{\omega} S({}_0D_t^{-\omega}F(t);\varphi) \tag{2.22}$$

which is further given as

$$\begin{aligned}&\bar{F}(\varphi)\\ &= F_0\int_0^\infty e^{-t}\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)(\varphi)^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(\varphi)^{\nu}}{\nu!}\right)dt\\ &\qquad\times\left(\frac{\Gamma(\mathrm{c}+\nu+1)}{\Gamma(\mathrm{c}+\nu+\gamma+1)}\times\varphi^{\gamma}t^{c+\nu+\gamma}\right)-\xi^{\omega}(\varphi)^{\omega}F(\varphi)\end{aligned} \tag{2.23}$$

Solved (2.23) by reordering the integration and summation, the expression is simplified as follows:

$$\begin{aligned}&\bar{F}(\varphi)[1+\xi^{\omega}(\varphi)^{\omega}]\\ &= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)\varphi^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(\varphi)^{\nu}}{\nu!}\right)\\ &\qquad\times\frac{\Gamma(\mathrm{c}+\nu+1)}{\Gamma(\mathrm{c}+\nu+\gamma+1)}\int_0^\infty e^{-t}\varphi^{\gamma}\,t^{c+\nu+\gamma}dt\end{aligned} \tag{2.24}$$

that is

$$\begin{aligned}&= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{1}{\nu!}\right)\\ &\qquad\times\frac{\Gamma(\mathrm{c}+\nu+1)}{\Gamma(\mathrm{c}+\nu+\gamma+1)}\times\varphi^{\mathrm{c}+\nu+\gamma}\Gamma(\mathrm{c}+\nu+\gamma+1)\end{aligned} \tag{2.25}$$

which leads to

$$\begin{aligned}\bar{F}(\varphi) &= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(1)^{\nu}}{\nu!}\right)\\ &\qquad\times\Gamma(\mathrm{c}+\nu+1)\;\times\sum_{\mu=0}^{\infty}(-1)^{\mu}(\xi\varphi)^{\mu\omega}\varphi^{\mathrm{c}+\nu+\gamma}\end{aligned} \tag{2.26}$$

Now, apply the Inverse Sumudu transform technique in (2.26) and use the property (2.7):

We obtain

$$\begin{aligned}F(t) &= F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)}{\Gamma(c+f+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(1)^{\nu}}{\nu!}\right)\\ &\qquad\times(1+\mathrm{c})_\nu\;\times\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}(\xi)^{\mu\omega}\frac{t^{\mu\omega+c+\gamma-1}}{\Gamma(\mu\omega+c+\nu+\gamma)}\right)\end{aligned} \tag{2.27}$$

Now, Equation (2.27) can be written as follows:

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}}\frac{(-1)^{\mathrm{c}}\Gamma(r+f+1)t^{c+\gamma}}{\Gamma(c+f+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(\mathrm{s}_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(\mathrm{r}_1)_\nu(\mathrm{r}_2)_\nu\ldots(\mathrm{r}_\mathrm{v})_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\frac{(t)^{\nu}}{\nu!}\right)$$

$$\times (1+\mathrm{c})_{\nu} \times \frac{1}{\mathrm{t}} E_{\omega,c+\nu+\gamma}(\xi t)^{\omega} \tag{2.28}$$

By using the specific values of parameters t and ω, the numerical analysis of the results can be studied through Table 3 and Figure 3 (i) and Figure 3 (ii) to obtain the graphical representation of the finding and behavior.

The values of $F(t)$ corresponding to various values of t and ω for equation (2.21)

t	$\mathbf{F(t)}$ **at** $\boldsymbol{\omega = 0.1}$	$\mathbf{F(t)}$ **at** $\boldsymbol{\omega = 0.5}$	$\mathbf{F(t)}$ **at** $\boldsymbol{\omega = 0.9}$	$\mathbf{F(t)}$ **at** $\boldsymbol{\omega = 1.3}$
4.0	−0.010336	−1.640386	−2.12676	−1.31529
4.2	0.0783	−1.989072	−2.997447	−2.021792
4.4	0.211542	−2.327885	−4.134579	−3.040457
4.6	0.4018	−2.6130617	−5.586967	−4.481118
4.8	0.663638	−2.7809	−7.40046	−6.482166
5.0	1.0139189	−2.7424401	−9.6117396	−9.2147267
5.2	1.4720524	−2.377222	−12.23948	−12.886301
5.4	2.0601939	−1.5260498	−15.27218	−17.743289
5.6	2.8034665	0.01736667	−18.651807	−24.0715455
5.8	3.7301898	2.5159746	−22.252248	−32.193846
6.0	4.8721086	6.3002678	−25.851573	−42.46273135
6.2	6.26464	11.780612	−29.09655	−55.24672
6.4	7.9471127	19.46112	−31.45809	−70.90733
6.6	9.963016	29.955252	−32.1757	−89.76354
6.8	12.360262	44.00311	−30.189058	−112.039522
7.0	15.191442	62.490833	−24.0543584	−137.790379
7.2	18.514099	86.4719514	−11.84288	−166.799279
7.4	22.39099	117.191019	8.9811598	−198.437945
7.6	26.890393	156.109613	41.7067235	−231.48055
7.8	32.086338	204.93485	90.549672	−263.85897
8.0	38.05894	265.650629	160.858199	−292.344837

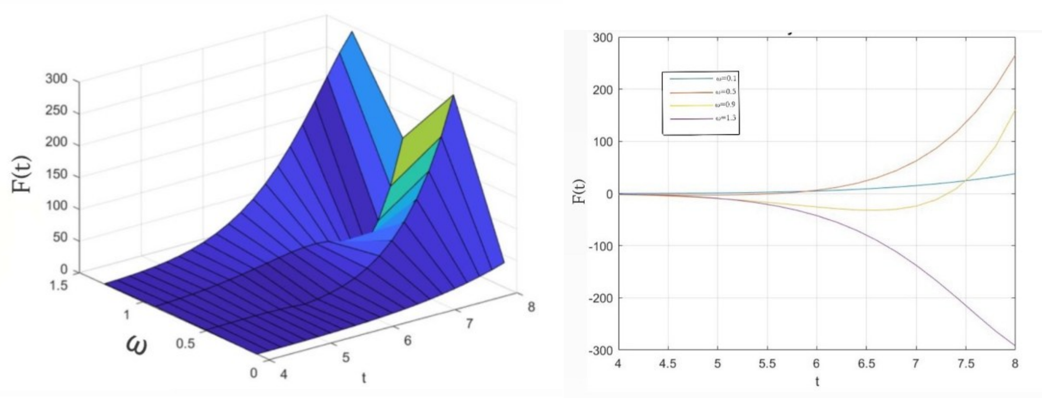

Figure 3 (a) 3D graph for (2.21) (b) 2D graph for (2.21)

Theorem 4: If $\omega > 0, \xi > 0, \gamma \neq \omega, \xi \neq \varepsilon$ *and* $r \in \nu, r-1 < \omega < r, 0 \leq t < \infty$ *and* $f > -1$, with the conditions of (1.13), then the FKE

$$F(t) - F_0 \left[{}_0D_t^{-\gamma} \left\{ L_r^f(\xi^{\omega} t^{\omega}) \mathrm{S}_{(\mathrm{u},\mathrm{v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_{\mathrm{v}}; \xi^{\omega} t^{\omega}) \right\} \right] = -\varepsilon^{\omega} {}_0D_t^{-\omega} F(t) \tag{2.29}$$

yielding the solution

$$F(t) = F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi)^{\omega c}\, t^{wc+\gamma}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi t)^{\omega\nu}}{\nu!}\right)$$

$$\times\,\Gamma(\omega(c+\nu)+1)\times\tfrac{1}{t}E_{\omega,\omega c+\omega\nu+\gamma}(-\varepsilon^{\omega}t^{\omega}) \qquad (2.30)$$

Proof: By employing the Sumudu Transform technique on (2.29) and incorporating (1.15), we obtain

$$S(F(t);\varphi) = F_0 S\left[{}_0D_t^{-\gamma}\left\{L_r^{f}(\xi^{\omega}t^{\omega})S_{(u,v)}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_v;\xi^{\omega}t^{\omega});\varphi\right\}\right]-\varepsilon^{\omega}S({}_0D_t^{-\omega}F(t);\varphi)$$
$$(2.31)$$

which is further given as
$\bar{F}(\varphi)$

$$= F_0\int_0^{\infty} e^{-t}\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)\left(\xi^{\omega}(\varphi)^{\omega}\right)^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{\left(\xi^{\omega}(\varphi)^{\omega}\right)^{\nu}}{\nu!}\right)$$
$$\times\left(\tfrac{\Gamma(\omega c+\omega\nu+1)}{\Gamma(\omega c+\omega\nu+\gamma+1)}\times\varphi^{\gamma}t^{\omega c+\omega\nu+\gamma+1}\right)dt-\varepsilon^{\omega}(\varphi)^{\omega}F(\varphi) \qquad (2.32)$$

Solved (2.32) by reordering the integration and summation, the expression is simplified as follows:

$\bar{F}(\varphi)[1+\varepsilon^{\omega}(\varphi)^{\omega}]$

$$= F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)\,(\xi^{\omega}\varphi^{\omega})^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega}\varphi^{\omega})^{\nu}}{\nu!}\right)$$
$$\times\frac{\Gamma(\omega c+\omega\nu+1)}{\Gamma(\omega c+\omega\nu+\gamma+1)}\times\int_0^{\infty} e^{-t}\,\varphi^{\gamma}t^{\omega(c+\nu)+\gamma+1}dt$$

$$= F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi^{\omega})^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega})^{\nu}}{\nu!}\right)$$

$$\times\tfrac{\Gamma(\omega c+\omega\nu+1)}{\Gamma(\omega c+\omega\nu+\gamma+1)}\times\varphi^{\omega c+\omega\nu+\gamma}\times\Gamma(\omega(c+\nu)+\gamma+1) \qquad (2.33)$$

which leads to

$$\bar{F}(\varphi) = F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi^{\omega})^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega})^{\nu}}{\nu!}\right)$$

$$\times\,\Gamma(\omega(c+\nu)+1)\quad\times\textstyle\sum_{\mu=0}^{\infty}(-1)^{\mu}\,(\varepsilon\varphi)^{\mu\omega}\varphi^{\omega c+\omega\nu+\gamma} \qquad (2.34)$$

Now, apply the Inverse Sumudu transform technique in (2.34) and use property (2.7):

We obtain

$$F(t) = F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)\xi^{\omega c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\ldots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\ldots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{\xi^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(c+\nu)+1) \quad \times \left(\sum_{\mu=0}^{\infty}(-1)^{\mu}\,(\varepsilon)^{\mu\omega}\,\frac{t^{\omega(\mu+c+\nu)+\gamma-1}}{\Gamma(\omega(\mu+c+\nu)+\gamma)}\right) \tag{2.35}$$

$$= F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi t)^{\omega c}t^{\gamma}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_{\nu}(s_2)_{\nu}\dots(s_u)_{\nu}(\rho)_{\nu\alpha,\kappa}}{(r_1)_{\nu}(r_2)_{\nu}\dots(r_v)_{\nu}\,\Gamma_{\kappa}(\nu\theta+\eta)}\;\frac{(\xi t)^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(c+\nu)+1) \quad \times \frac{1}{t}\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}\,\frac{(\varepsilon t)^{\mu\omega}}{\Gamma(\omega(\mu+c+\nu)+\gamma)}\right) \tag{2.36}$$

Now, Equation (2.36) can be written as follows:

$$F(t) = F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi)^{\omega c}\;t^{wc+\gamma}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_{\nu}(s_2)_{\nu}\dots(s_u)_{\nu}(\rho)_{\nu\alpha,\kappa}}{(r_1)_{\nu}(r_2)_{\nu}\dots(r_v)_{\nu}\,\Gamma_{\kappa}(\nu\theta+\eta)}\;\frac{(\xi t)^{\omega\nu}}{\nu!}\right)$$

$$\times \Gamma(\omega(c+\nu)+1) \times \frac{1}{t}E_{\omega,\omega c+\omega\nu+\gamma}(-\varepsilon^{\omega}t^{\omega}) \tag{2.37}$$

By using the specific values of parameters t and ω, the numerical analysis of the results can be studied through Table 4 and Figure 4 (i) and Figure 2 (ii) to obtain the graphical representation of the finding and behavior.

The values of $F(t)$ corresponding to various values of t and ω for equation (2.30)

t	$\mathbf{F(t)\ at\ \omega = 0.1}$	$\mathbf{F(t)\ at\ \omega = 0.5}$	$\mathbf{F(t)\ at\ \omega = 0.9}$	$\mathbf{F(t)\ at\ \omega = 1.3}$
4.0	0.0001036	0.018719	0.1091	-2.0853347
4.2	0.00011	0.025632	0.2360249	-2.895723
4.4	0.000117	0.0344441	0.45531	-3.60429
4.6	0.000123	0.045529	0.8174796	-3.676066
4.8	0.0001304	0.0593	1.394738	-1.98866
5.0	0.0001371	0.07628351	2.2884427	3.6134706
5.2	0.0001439	0.096969	3.638475	17.032166
5.4	0.00015	0.121975	5.635046	45.0273832
5.6	0.000157	0.151965	8.533347	98.91327
5.8	0.0001643	0.1876789	12.6716	197.095354
6.0	0.000171	0.229926	18.49314	368.80066
6.2	0.000177	0.279598	26.57311	659.47633
6.4	0.000184	0.33767	37.65071	1138.49215
6.6	0.00019	0.4052	52.66769	1909.98729
6.8	0.000198545	0.48335	72.81418	3127.96143
7.0	0.000205	0.573388	99.582832	5017.03762
7.2	0.000212	0.676487	134.83256	7900.73317
7.4	0.000219	0.7946064	180.8631	12239.58143
7.6	0.0002261	0.92884	240.50184	18682.07257
7.8	0.000233	1.081046	317.20464	28132.1456
8.0	0.0002399	1.253045	415.17209	41837.89496

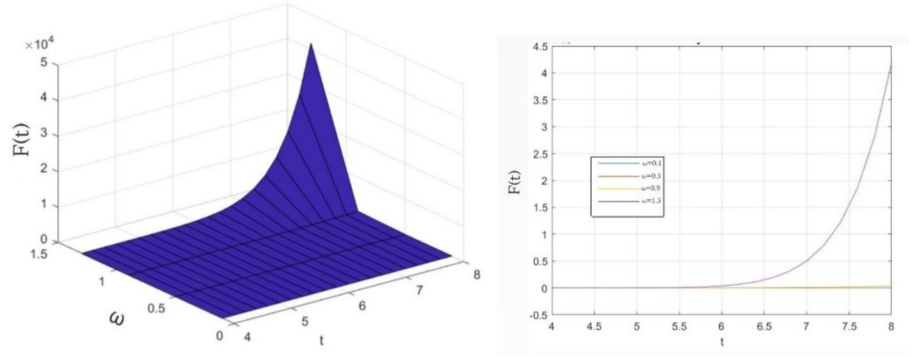

Figure 4 **(a) 3D graph for (2.30)** **(b) 2D graph for (2.30)**

3. Special Cases

(i) The following results are obtained from Theorems (1) and (2) when we choose $f = 0$

Corollary 1. Let $\omega > 0, \xi > 0$ *and* $r \in \nu, r-1 < \omega < r, 0 \le t < \infty$ *and* $f > -1$, the equation reduces as

$$F(t) - F_0\left\{L_r(t)\mathrm{S}_{(\mathrm{u,v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_{\mathrm{v}}; t)\right\} = -\xi^{\omega}\,{}_0D_t^{-\omega}F(t) \tag{3.1}$$

and provides the solution

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+1)t^c}{[\Gamma(c+1)]^2\Gamma(r-c+1)} \sum_{\mathrm{\nu}=0}^{\infty} \frac{(\mathrm{s}_1)_{\mathrm{\nu}}(s_2)_{\mathrm{\nu}} \dots (s_u)_{\mathrm{\nu}}(\rho)_{\mathrm{\nu}\alpha,\kappa}}{(\mathrm{r}_1)_{\mathrm{\nu}}(\mathrm{r}_2)_{\mathrm{\nu}} \dots (\mathrm{r}_{\mathrm{v}})_{\mathrm{\nu}}\, \Gamma_\kappa(\mathrm{\nu}\theta+\eta)} \frac{(t)^{\mathrm{\nu}}}{\mathrm{\nu}!}\right)$$

$$\times\, \Gamma(\mathrm{c}+\mathrm{\nu}+1) \times \frac{1}{\mathrm{t}} E_{\omega, c+\mathrm{\nu}}(\xi t)^{\mu\omega} \tag{3.2}$$

Corollary 2: If $\varepsilon > 0, \omega > 0, > 0, \xi \neq \varepsilon$ *and* $r \in \nu, r-1 < \omega < r, 0 \le t < \infty$ *and* $f > -1$, the equation reduces as

$$F(t) - F_0\left\{L_r(\xi^\omega t^\omega)\mathrm{S}_{(\mathrm{u,v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_{\mathrm{v}}; \xi^\omega t^\omega)\right\} = -\varepsilon^{\omega}\,{}_0D_t^{-\omega}F(t) \tag{3.3}$$

and provides the solution

$$F(t) = F_0\left(\sum_{c=0}^{\mathrm{r}} \frac{(-1)^{\mathrm{c}}\Gamma(r+1)(\xi^\omega t^\omega)^c}{[\Gamma(c+1)]^2\Gamma(r-c+1)} \sum_{\mathrm{\nu}=0}^{\infty} \frac{(\mathrm{s}_1)_{\mathrm{\nu}}(s_2)_{\mathrm{\nu}} \dots (s_u)_{\mathrm{\nu}}(\rho)_{\mathrm{\nu}\alpha,\kappa}}{(\mathrm{r}_1)_{\mathrm{\nu}}(\mathrm{r}_2)_{\mathrm{\nu}} \dots (\mathrm{r}_{\mathrm{v}})_{\mathrm{\nu}}\, \Gamma_\kappa(\mathrm{\nu}\theta+\eta)} \frac{(\xi^\omega t^\omega)^{\mathrm{\nu}}}{\mathrm{\nu}!}\right)$$

$$\times\, \Gamma(\omega(\mathrm{c}+\mathrm{\nu})+1) \times \frac{1}{\mathrm{t}} E_{\omega, \omega c+\omega\mathrm{\nu}}(-\varepsilon^\omega t^\omega) \tag{3.4}$$

(ii) The following results are obtained from Theorems (3) and (4) when we choose $\gamma = \omega$

Corollary 3. Let $\omega > 0, \xi > 0$ *and* $r \in \nu, r-1 < \omega < r, 0 \le t < \infty$ *and* $f > -1$, the equation reduces as

$$F(t) - F_0\left[{}_0D_t^{-\omega}\left\{L_r^f(t^\omega)\mathrm{S}_{(\mathrm{u,v})}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1, s_2, .. s_u; r_1, r_2, .. r_{\mathrm{v}}; t^\omega)\right\}\right] = -\xi^{\omega}\,{}_0D_t^{-\omega}F(t) \tag{3.5}$$

and provides the solution

$$F(t)=F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+1)t^{c}}{\Gamma(c+f+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\dots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\dots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(t)^{\nu}}{\nu!}\right)$$

$$\times(1+c)_\nu\times\frac{1}{t}E_{\omega,c+\nu+\omega(-}\xi^{\omega}t^{\omega})\qquad(3.6)$$

Corollary 4. Let $\omega>0,\xi>0,\xi\neq\varepsilon\ and\ r\in\nu,r-1<\omega<r,0\leq t<\infty\ and\ f>-1$, the equation reduces as

$$F(t)-F_0\left[{}_0D_t^{-\omega}\left\{L_r^{f}(\xi^{\omega}t^{\omega})S_{(u,v)}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_v;\xi^{\omega}t^{\omega})\right\}\right]=-\varepsilon^{\omega}{}_0D_t^{-\omega}F(t)\qquad(3.7)$$

and provides the solution

$$F(t)=F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi^{\omega}t^{\omega})^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\dots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\dots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega}t^{\omega})^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\omega(c+\nu)+1)\times\frac{1}{t}E_{\omega,\omega c+\omega\nu+\omega}(-\varepsilon^{\omega}t^{\omega})\qquad(3.8)$$

(iii) The following results are obtained from Theorems (2) when we choose $\varepsilon=\xi$

Corollary 5. Let $\omega>0,\xi>0\ and\ r\in\nu,r-1<\omega<r,0\leq t<\infty\ and\ f>-1$, the equation reduces as

$$F(t)-F_0\left\{L_r^{f}(\xi^{\omega}t^{\omega})S_{(u,v)}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_v;\xi^{\omega}t^{\omega})\right\}=-\xi^{\omega}{}_0D_t^{-\omega}F(t)\qquad(3.9)$$

and provides the solution

$$F(t)=F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+1)(\xi^{\omega}t^{\omega})^{c}}{[\Gamma(c+1)]^{2}\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\dots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\dots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega}t^{\omega})^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\omega(c+\nu)+1)\times\frac{1}{t}E_{\omega,\omega c+\omega\nu}(-\xi^{\omega}t^{\omega})\qquad(3.10)$$

(iv) The following results are obtained from Theorems (4) when we choose $\varepsilon=\xi$ and $\gamma=\omega$

Corollary 6. Let $\omega>0,\xi>0\ and\ r\in\nu,r-1<\omega<r,0\leq t<\infty\ and\ f>-1$, the equation reduces as

$$F(t)-F_0\left[{}_0D_t^{-\omega}\left\{L_r^{f}(\xi^{\omega}t^{\omega})S_{(u,v)}^{(\theta,\eta,\rho,\alpha,\kappa)}(s_1,s_2,..s_u;r_1,r_2,..r_v;\xi^{\omega}t^{\omega})\right\}\right]=-\xi^{\omega}{}_0D_t^{-\omega}F(t)\qquad(3.11)$$

and provides the solution

$$F(t)=F_0\left(\sum_{c=0}^{r}\frac{(-1)^{c}\Gamma(r+f+1)(\xi^{\omega}t^{\omega})^{c}}{\Gamma(c+f+1)\Gamma(c+1)\Gamma(r-c+1)}\sum_{\nu=0}^{\infty}\frac{(s_1)_\nu(s_2)_\nu\dots(s_u)_\nu(\rho)_{\nu\alpha,\kappa}}{(r_1)_\nu(r_2)_\nu\dots(r_v)_\nu\,\Gamma_\kappa(\nu\theta+\eta)}\,\frac{(\xi^{\omega}t^{\omega})^{\nu}}{\nu!}\right)$$

$$\times\Gamma(\omega(c+\nu)+1)\times\frac{1}{t}E_{\omega,\omega c+\omega\nu+\omega}(-\xi^{\omega}t^{\omega})\qquad(3.12`)$$

4. Graphical Interpretation of Results

Here, we employ a variety of suitable parameter assignments to give a graphical representation of the findings from Theorems 1 to 4. For every graph, we provide four outputs based on changing the parameter values, which are presented as the solution of the equation. We plot four graphs of equation (2.2) in Figure 1 (a) and Figure 1 (b), equation (2.12) in Figure 2 (a) and Figure 2 (b), equation (2.21) in Figure 3 (a) and Figure 3 (b), and equation (2.30) in Figure 4 (a) and Figure 4 (b). Figure 1 (a) and Figure 1 (b) are the graphical solutions of Theorem 1 using the assigned value to the parameters $F_0 = 1, m = 1, \xi = 1, t = 4, \nu = 0.1, 0.5, 0.9, 1.3$. Similarly, the graphical solutions for Theorems (2), (3), and (4) are shown in Figures 2 (a) and 2 (b), Figures 3 (a) and 3 (b), and 4 (a) and 4 (b), respectively.

By analyzing these figures, we can see that $F(t) > 0$ for $t > 0$ and $F(t) \to \infty$ as $t \to \infty$ which shows that the solution is stable and converges. The parameters can be adjusted to any value, which allows the reader to plot additional graphs and find particular solutions to the kinetic equations. Choosing the numerical values for the parameter assignments is predicated on the idea that a wide variety of scenarios will be available to examine how the solutions behave under various circumstances. We hope to investigate the effects of parameters like m and ω on the solutions of the kinetic equations and spot any patterns or trends that show up by methodically altering these variables.

5. Conclusion

In this study, we propose to solve generalized fractional kinetic equations using Srivastava and Laguerre polynomials by using the Sumudu transform approach. The Mittag-Leffler function has been used to express the results that were obtained. Furthermore, by using MATLAB for graphical representations, we have not only published these solutions but also convincingly illustrated their behavior under a variety of parametric conditions. The solutions to all four equations (2.1), (2.11), (2.20), and (2.29), as interpreted graphically, continue to be non-negative. Moreover, observing these figures $\digamma(t)>0$ for $t>0$ and $F(t) > 0$ for $t > 0$ and $F(t) \to \infty$ as $t \to \infty$, we can say that the solution is convergent and stable.\\

This study explores the fractional kinetic equation involving Laguerre polynomials and their corresponding fractional derivatives. It highlights the significant influence of the derivative order on the system ρ behavior and demonstrates, through the application of the Sumudu transform method, how changes in this order affect the nature of the solutions. It is possible to observe how different orders affect the system ρ patterns, stability, and overall dynamics, which sheds light on this parameter's significance in the investigation context.

In addition to being flexible, the investigation's conclusions have the potential to expand on existing knowledge in this field and pave the way for discoveries. The application of Laguerre polynomials and fractional derivatives to fractional kinetic equations is an intriguing avenue for further study and application. This work lays the groundwork for future research that will deepen our understanding of fractional calculus and its uses.

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