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Twain Secure Perfect Dominating Sets and Twain Secure Perfect Domination Polynomials of Centipedes

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Abstract: Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A subset S of V is called a twain secure perfect dominating set of G (TSPD-set) if every vertex $v \in V \setminus S$ is adjacent to exactly one vertex $u \in S$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The twain secure perfect domination number of G, represented as $\gamma_{tsp}(G)$, is the lowest cardinality of a twain secure perfect dominating set of G. The centipede with $2n$ vertices is represented by P_n^*, and the family of all twain secure perfect dominating sets of P_n^* with cardinality i is represented by $D_{tsp}(P_n^*, i)$. Let $d_{tsp}(P_n^*, i) = |D_{tsp}(P_n^*, i)|$. This article builds $D_{tsp}(P_n^*, i)$ and derives a recursive formula for $d_{tsp}(P_n^*, i)$. With this recursive formula, we examine the polynomial $D_{tsp}(P_n^*, x) = \sum_{i=n}^{2n} d_{tsp}(P_n^*, i)\, x^i$ which we call the twain secure perfect domination polynomial of centipedes. To create all twain secure perfect dominating sets of centipedes and twain secure perfect domination polynomials of centipedes, we employ a recursive approach in this study.

Keywords and Phrases: centipede, twain secure perfect dominating set, twain secure perfect domination number, twain secure perfect domination polynomial.

Mathematics Subject Classification: 05C69, 05C31

1. Introduction

A finite undirected connected graph without loops or multiple edges is referred to as a graph $= (V, E)$. G's order and size are shown by the numbers n and m, respectively. For fundamental terms and definitions, see [3]. Consider two vertices u and v. If uv is one of G's edges, then u and v are considered adjacent. A vertex v in a graph G has an open neighborhood defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and a closed neighborhood defined as $N_G[v] = N_G(v) \cup \{v\}$. A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G) \setminus S$ is adjacent to a vertex $u \in S$. The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a $\gamma -$ set of G [1]. A dominating set S is called a secure dominating set if each $v \in V(G) \setminus S$ there exists $u \in N(v) \cap S$ such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set of G. Cockayne et al was introduced the concept of secure domination of graphs [4]. A dominating set S is called a perfect dominating set if every vertex in $V(G) \setminus S$ is adjacent to exactly one vertex in S. The perfect domination number $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G. The concept of perfect domination of graphs was introduced by Weichsel [11]. We introduce the concept of twain secure perfect domination of centipedes in this work. A dominating set S is called a twain secure perfect dominating set of G (TSPD-set) if every vertex $v \in V(G) \setminus S$ is adjacent to exactly one vertex $u \in S$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The twain secure perfect domination number of G, represented as $\gamma_{tsp}(G)$, is the lowest cardinality of a twain secure perfect dominating set of G. A simple path, represented by P_n is one in which all of its internal vertices have degree two and its end vertices have degree one. The centipede with $2n$ vertices, represented by P_n^*, is created by adding a single pendant edge to each vertex of a path P_n. We cite S. Alikhani and Y.H. Peng [2] for the definition of centipede. $\gamma_{tsp}(P_n^*)$ is the twain secure perfect domination number of P_n^*. The family of all twain secure perfect dominating sets of P_n^* with cardinality i is denoted by $D_{tsp}(P_n^*, i)$. $d_{tsp}(P_n^*, i) = |D_{tsp}(P_n^*, i)|$, let's say. The twain secure perfect domination polynomial of P_n^* is thus $D_{tsp}(P_n^*, x) = \sum_{i=n}^{2n} d_{tsp}(P_n^*, i)\, x^i$.

Throughout this work, we use $[2n]$ to indicate the set $\{1, 2, \ldots, 2n\}$ and $\{1, 2, \ldots, 2n-1\}$ by $[2n-1]$. In order to analyze twain secure perfect dominating sets of centipedes, we must first examine twain secure perfect dominating sets and then the twain secure perfect domination polynomial of $P_n^* - \{2n\}$.

2. Twain Secure Perfect Dominating Sets and Twain Secure Perfect Domination Polynomials of $P_n^* - \{2n\}$

Lemma 2.1. For every $n \in \mathbb{N}$,

(i) $\gamma_{tsp}(P_n^*) = n$.
(ii) $\gamma_{tsp}(P_n^* - \{2n\}) = n$.
(iii) $D_{tsp}(P_n^*, i) = \emptyset$ if and only if $i < n$ or $i > 2n$.
(iv) $D_{tsp}(P_n^* - \{2n\}, i) = \emptyset$ if and only if $i < n$ or $i > 2n-1$.

Lemma 2.2. For every $n \in \mathbb{N}$,

(i) If $D_{tsp}(P_{n-1}^*, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$, then $D_{tsp}(P_n^* - \{2n\}, i) \neq \emptyset$.
(ii) If $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$, then $D_{tsp}(P_n^* - \{2n\}, i) = \emptyset$.
(iii) If $D_{tsp}(P_{n-1}^*, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$, then $D_{tsp}(P_n^* - \{2n\}, i) \neq \emptyset$.

Proof:

(i) The assumptions are that $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) = \emptyset$. Lemma 2.1(iii) tells us that $n - 1 \leq i - 1 \leq 2n - 2$ and $i - 2 < n - 2$ or $i - 2 > 2n - 4$. Therefore, $n \leq i \leq 2n - 1, n - 1 \leq i - 1 \leq 2n - 2$. According to Lemma 2.1(iv), $D_{tsp}(P_n^* - \{2n\}, i) \neq \emptyset$.

(ii) Assume that $D_{tsp}(P_{n-1}^*, i - 1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) = \emptyset$. $i - 1 < n - 1$ or $i - 1 > 2n - 2$ and $i - 2 < n - 2$ or $i - 2 > 2n - 4$ according to Lemma 2.1(iii). Which results in $i - 1 > 2n - 2$ or $i - 2 < n - 2$. Consequently, either $i < n$ or $i > 2n - 1$. $D_{tsp}(P_n^* - \{2n\}, i) = \emptyset$ according to Lemma 2.1(iv).

(iii) $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) \neq \emptyset$, assuming such. Lemma 2.1(iii)
determines that $n - 1 \leq i - 1 \leq 2n - 2$ and $n - 2 \leq i - 2 \leq 2n - 4$. Which yields $n \leq i$ and $i \leq 2n - 1$. Thus $n \leq i \leq 2n - 1$. $D_{tsp}(P_n^* - \{2n\}, i) \neq \emptyset$ according to Lemma 2.1(iv).

Lemma 2.3. For every $n \geq 3$ and $D_{tsp}(P_n^* - \{2n\}, i) \neq \emptyset$,

(i) $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) = \emptyset$ if and only if $i = 2n - 1$.

(ii) $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) \neq \emptyset$ if and only if $n \leq i \leq 2n - 2$.

Proof:

(i) $i - 2 < n - 2$ or $i - 2 > 2n - 4$ according to Lemma 2.1(iii), since $D_{tsp}(P_{n-2}^*, i - 2) = \emptyset$. If $i - 2 < n - 2$, then $i < n$. Lemma 2.1(iv) thus states that $D_{tsp}(P_n^* - \{2n\}, i) = \emptyset$. This is contradictory. Consequently, $i - 2 > 2n - 4$. Which gives,

$$i \geq 2n - 1. \quad (1)$$

Because $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$, Lemma 2.1(iii) implies that $n - 1 \leq i - 1 \leq 2n - 2$. Which gives,

$$i \leq 2n - 1. \quad (2)$$

From (1) and (2), $i = 2n - 1$.

Conversely, assume that $i = 2n - 1$. This implies that $i - 1 = 2n - 2$. Following the Lemma 2.1(iii), $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$. Since $i = 2n - 1$. This yields $i - 2 > 2n - 4$. Using the Lemma 2.1(iii), the set $D_{tsp}(P_{n-2}^*, i - 2)$ is empty.

(ii) Assume $D_{tsp}(P_{n-1}^*, i - 1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) \neq \emptyset$. Lemma 2.1(iii) indicates that $n - 1 \leq i - 1 \leq 2n - 2$ and $n - 2 \leq i - 2 \leq 2n - 4$. Combining the aforementioned, $n - 1 \leq i - 1 \leq 2n - 3$. This implies that $n \leq i \leq 2n - 2$.

Conversely, assume that $n \leq i \leq 2n - 2$. Therefore, $\gamma_{tsp}(P_{n-1}^*) = n - 1 \leq$

$i-1 \leq 2n-3 \leq 2n-2$. So, we have $D_{tsp}(P_{n-1}^{*}, i-1) \neq \emptyset$ by Lemma 2.1(iii). Similarly, $D_{tsp}(P_{n-2}^{*}, i-2) \neq \emptyset$.

Theorem 2.4. In case $i = n$ and for every $n \geq 3$, then $D_{tsp}(P_{n}^{*} - \{2n\}, i) = \{\{1,3,5,7,9,\ldots,2n-3,2n-1\},\{1,3,5,7,9,\ldots,2n-3,2n-2\}\}$.

Proof:

Since $i = n$, the sets $\{1,3,5,7,9,\ldots,2n-3,2n-1\}$ and $\{1,3,5,7,9,\ldots,2n-3,2n-2\}$ have n elements each. By the definition of twain secure perfect domination of $P_{n}^{*} - \{2n\}$, $1,3,4$ cover all the vertices up to 5 and $1,3,5$ cover all the vertices up to 5 for $n = 3$, respectively. Continuing this way, we get that $\{1,3,5,7,9,\ldots,2n-3,2n-1\}$ and $\{1,3,5,7,9,\ldots,2n-3,2n-2\}$ cover all the vertices up to n. As a result, $\{1,3,5,7,9,\ldots,2n-3,2n-1\}$ and $\{1,3,5,7,9,\ldots,2n-3,2n-2\}$ are twain secure perfect dominating sets.

Theorem 2.5. For every $n \geq 3$ and $i \geq n+1$,

(i) If $D_{tsp}(P_{n-1}^{*}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^{*}, i-2) = \emptyset$, then $D_{tsp}(P_{n}^{*} - \{2n\}, i) = \{[2n-1]\}$.

(ii) If $D_{tsp}(P_{n-1}^{*}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^{*}, i-2) \neq \emptyset$, then $D_{tsp}(P_{n}^{*} - \{2n\}, i) = \{\{X \cup \{2n-2\}$ if X ends with $2n-3\} \cup \{X \cup \{2n-1\}$ if X ends with $2n-2\} \cup \{Y \cup \{2n-3, 2n-1\}\}$, where $X \in D_{tsp}(P_{n-1}^{*}, i-1), Y \in D_{tsp}(P_{n-2}^{*}, i-2)$.

Proof:

(i) Lemma 2.3(i) states that $i = 2n-1$, since $D_{tsp}(P_{n-1}^{*}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^{*}, i-2) = \emptyset$. $D_{tsp}(P_{n}^{*} - \{2n\}, i) = \{[2n-1]\}$, as a result.

(ii) The construction of $D_{tsp}(P_{n}^{*} - \{2n\}, i)$ is derived from $D_{tsp}(P_{n-1}^{*}, i-1)$ and $D_{tsp}(P_{n-2}^{*}, i-2)$. Assume that X is the twain secure perfect dominating set of P_{n-1}^{*} with cardinality $i-1$. The elements of $D_{tsp}(P_{n-1}^{*}, i-1)$ terminate in $2n-3$ or $2n-2$.

- If $2n-3 \in X$, then the elements of $D_{tsp}(P_{n-1}^{*}, i-1)$ belong to $D_{tsp}(P_{n}^{*} - \{2n\}, i)$ by adjoining $2n-2$.
- If $2n-2 \in X$, then the elements of $D_{tsp}(P_{n-1}^{*}, i-1)$ belong to $D_{tsp}(P_{n}^{*} - \{2n\}, i)$ by adjoining $2n-1$.

Let Y be the twain secure perfect dominating set of P_{n-2}^{*} with cardinality $i-2$. By adjoining $2n-3$ and $2n-1$, every elements of $D_{tsp}(P_{n-2}^{*}, i-2)$ belong to $D_{tsp}(P_{n}^{*} - \{2n\}, i)$.

Thus $\{\{X \cup \{2n-2\}$ if X ends with $2n-3\} \cup \{X \cup \{2n-1\}$ if X ends with $2n-2\} \cup \{Y \cup \{2n-3, 2n-1\}\} \subseteq$ to $D_{tsp}(P_{n}^{*} - \{2n\}, i)$. (3)

Conversely, assume that $Z = D_{tsp}(P_n^* - \{2n\}, i)$. Here, $2n - 1$ or $2n - 2$ is the end vertex of each element in $D_{tsp}(P_n^* - \{2n\}, i)$.

- If $2n - 1 \notin Z$ and $2n - 2 \in Z$. Afterward, at least one vertex with the label $2n - 3$. Let's say $2n - 3 \in Z$. Then $Z = X \cup \{2n - 2\}$, for some $X \in D_{tsp}(P_{n-1}^*, i - 1)$.
- If $2n - 1 \in Z$. Then at least one vertex with the labels $2n - 2$ or $2n - 3$. Suppose $2n - 2 \in Z$. Then $Z = X \cup \{2n - 1\}$, for some $X \in D_{tsp}(P_{n-1}^*, i - 1)$. Suppose $2n - 3 \in Z$ and $2n - 2 \notin Z$. Then $Z = Y \cup \{2n - 3, 2n - 1\}$, for some $Y \in D_{tsp}(P_{n-2}^*, i - 2)$.

Thus $D_{tsp}(P_n^* - \{2n\}, i) \subseteq \{\{X \cup \{2n-2\}$ if X ends with $2n-3\} \cup \{X \cup \{2n - 1\}$ if X ends with $2n - 2\} \cup \{Y \cup \{2n - 3, 2n - 1\}\}$. (4)

From (3) and (4), we have $D_{tsp}(P_n^* - \{2n\}, i) = \{\{X \cup \{2n - 2\}$ if X ends with $2n - 3\} \cup \{X \cup \{2n - 1\}$ if X ends with $2n - 2\} \cup \{Y \cup \{2n - 3, 2n - 1\}\}$, where $X \in D_{tsp}(P_{n-1}^*, i - 1), Y \in D_{tsp}(P_{n-2}^*, i - 2)$.

Theorem 2.6. Let $D_{tsp}(P_n^* - \{2n\}, i)$ be the family of all twain secure perfect dominating sets with cardinality i and $d_{tsp}(P_n^* - \{2n\}, i) = |D_{tsp}(P_n^* - \{2n\}, i)|$. For every $n \geq 3$,

(i) If $i = n$, then $d_{tsp}(P_n^* - \{2n\}, i) = d_{tsp}(P_{n-1}^*, i-1) + d_{tsp}(P_{n-2}^*, i-2) - 2$.

(ii) If $i \geq n + 1$, then $d_{tsp}(P_n^* - \{2n\}, i) = d_{tsp}(P_{n-1}^*, i-1) + d_{tsp}(P_{n-2}^*, i-2)$.

With initial values, $D_{tsp}(P_1^*, x) = 2x + x^2, D_{tsp}(P_2^*, x) = 2x^2 + 2x^3 + x^4$.

Proof:

(i) Proof follows from Theorem 2.4.

(ii) Proof follows from Theorem 2.5(ii).

3. Twain Secure Perfect Dominating Sets and Twain Secure Perfect Domination Polynomials of Centipedes ($\boldsymbol{P_n^*}$)

In this section, we investigate twain secure perfect dominating sets and twain secure perfect domination polynomials of centipedes.

Lemma 3.1.

(i) If $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-1}^*, i - 1) = \emptyset$, then $D_{tsp}(P_n^*, i) \neq \emptyset$.

(ii) If $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) \neq \emptyset$, then $D_{tsp}(P_n^*, i) \neq \emptyset$.

(iii) If $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) = \emptyset$, then $D_{tsp}(P_n^*, i) = \emptyset$.

(iv) If $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i - 2) \neq \emptyset$, then $D_{tsp}(P_n^*, i) \neq \emptyset$.

Proof:

(i) Since $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$, by Lemma 2.1, $n \leq i-1 \leq 2n-1$ and $i-1 < n-1$ or $i-1 > 2n-2$. Since $n \leq i-1 \leq 2n-1, n+1 \leq i \leq 2n$. Moreover, $n < n+1$. Therefore, $n \leq i \leq 2n$. Hence $D_{tsp}(P_n^*, i) \neq \emptyset$.

(ii) Because $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$, by Lemma 2.1, $i-1 < n$ or $i-1 > 2n-1$, and $n-2 \leq i-2 \leq 2n-4$. This yields $n \leq i \leq 2n-2 < 2n$. So, $n \leq i \leq 2n$. Hence, $D_{tsp}(P_n^*, i) \neq \emptyset$.

(iii) Since $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$, by Lemma 2.1, $i-1 < n$ or $i-1 > 2n-1$ and $i-2 < n-2$ or $i-2 > 2n-4$. This results in $i-2 < n-2$ or $i-1 > 2n-1$. This implies that either $i < n$ or $i > 2n$. Thus, $D_{tsp}(P_n^*, i) = \emptyset$.

(iv) By Lemma 2.1, $n \leq i-1 \leq 2n-1$ and $n-2 \leq i-2 \leq 2n-2$, since $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$. By combining the above two inequalities, we obtain $n-2 \leq i-2 \leq 2n-2$. This indicates that
$n \leq i \leq 2n$. Consequently, $D_{tsp}(P_n^*, i) \neq \emptyset$.

Lemma 3.2. If $D_{tsp}(P_n^*, i) \neq \emptyset$, for every $n \geq 3$,

(i) $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$, $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$ if and only if $i = 2n$.

(ii) $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$ if and only if $i = n$.

(iii) $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$ if and only if $n+1 \leq i \leq 2n-2$.

Proof:

(i) Since$D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$, by Lemma 2.1(iv), $n \leq i-1 \leq 2n-1$. Which implies,

$$i \leq 2n. \tag{5}$$

Given that $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$, by Lemma 2.1(iii), we get $i-1 < n-1$ or $i-1 > 2n-2$ and i – 2 < n – 2 or i – 2 > 2n – 4. Combining the preceding inequalities yield $i-2 < n-2$ or $i-1 > 2n-2$. If $i-2 < n-2$, then $i < n$. So, $D_{tsp}(P_n^*, i) = \emptyset$. Which is a paradox. If $i-1 > 2n-2$, then

$$i > 2n-1. \tag{6}$$

Combining (5) and (6) generates $2n-1 < i \leq 2n$. This means $2n \leq i \leq 2n$. So, $i = 2n$.

In contrast, suppose $i = 2n$. Which implies that $i-1 = 2n-1 > 2n-2$.

Lemma 2.1(iii) states that $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$. Likewise, $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$. Since $i = 2n$, $i-1 = 2n-1$. As per Lemma 2.1(iv), $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$.

(ii) Lemma 2.1(iii) demonstrates that $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$, then $i-1 < n$ or $i-1 > 2n-1$. As $i-1 < n, i < n+1$. It gives

$$i \leq n. \quad (7)$$

Since $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$, by Lemma 2.1(iv), $n-2 \leq i-2 \leq 2n-4$. That provides

$$n \leq i. \quad (8)$$

We can get $i = n$ by combining (7) and (8).

Conversely, assume that $i = n$. This implies that $i-1 < n$. So, $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$. Since $i = n$, we have $i-2 = n-2$. Consequently, $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$.

(iii) Lemma 2.1 states that $n \leq i-1 \leq 2n-1$ and $n-2 \leq i-2 \leq 2n-4$, since $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$. When the two inequalities are combined, we have $n-1 \leq i-2 \leq 2n-4$. This suggests that $n+1 \leq i \leq 2n-2$.
The converse is obvious.

Theorem 3.3. For every $n \geq 3$,

(i) If $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$ then $D_{tsp}(P_n^*, i) = \{\{1,3,5,7,\ldots,2n-3,2n-1\},\{2,4,6,8,\ldots,2n-2,2n\}\}$.

(ii) If $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$, $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$ then $D_{tsp}(P_n^*, i) = \{[2n]\}$.

(iii) If $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$ then $D_{tsp}(P_n^*, i)$ ={{ {{X ∪ {2n − 1} if X ends with $2n-2$} ∪ {X ∪ $\{2n\}$ if X ends with $2n-1$} ∪{Y ∪ {2n − 3, 2n − 1}}, where $X \in D_{tsp}(P_n^* - \{2n\}, i-1)$ and $Y \in D_{tsp}(P_{n-2}^*, i-2)$.

Proof:

(i) According to Lemma 3.2(ii), $i = n$, since $D_{tsp}(P_n^* - \{2n\}, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) \neq \emptyset$. The sets $\{1,3,5,7,\ldots,2n-3,2n-1\}$ and $\{2,4,6,8,\ldots,2n-2,2n\}$ clearly contain n members. According to the definition of twain secure perfect domination of P_n^*, $1,3,5,7,9$ cover all vertices up to 10 and $2,4,6,8,10$ cover all vertices up to 10 for $n = 5$. Using this method, we may conclude that $\{1,3,5,7,\ldots,2n-3,2n-1\}$ and $\{2,4,6,8,\ldots,2n-2,2n\}$ cover all vertices up to $2n$. The remaining sets with cardinality n do not have twain secure perfect dominating sets. Both $\{1,3,5,7,\ldots,2n-3,2n-1\}$ and $\{2,4,6,8,\ldots,2n-2,2n\}$ are twain secure perfect dominating sets.

(ii) Since $D_{tsp}(P_n^* - \{2n\}, i-1) \neq \emptyset$, $D_{tsp}(P_{n-1}^*, i-1) = \emptyset$ and $D_{tsp}(P_{n-2}^*, i-2) = \emptyset$, by Lemma 3.2(i), $i = 2n$. Therefore, $D_{tsp}(P_n^*, i) = \{[2n]\}$.

(iii) The construction of $D_{tsp}(P_n^*, i)$ follows from $D_{tsp}(P_n^* - \{2n\}, i-1)$ and $D_{tsp}(P_{n-2}^*, i-2)$. Let X be a twain secure perfect dominating set of $P_n^* - \{2n\}$ with cardinality $i-1$. The elements of $D_{tsp}(P_n^* - \{2n\}, i-1)$ end with $2n-2$ or
$2n-1$.

- If $2n-2 \in X, 2n-1 \notin X$, then the elements of $D_{tsp}(P_n^* - \{2n\}, i-1)$ belong to $D_{tsp}(P_n^*, i)$ by adjoining $2n-1$.

- If $2n-1 \in X$, then the elements of $D_{tsp}(P_n^* - \{2n\}, i-1)$ belong to $D_{tsp}(P_n^*, i)$ by adjoining $2n$.

Let Y be a twain secure perfect dominating set of P_{n-2}^* with cardinality $i-2$. All the elements of $D_{tsp}(P_{n-2}^*, i-2)$ belong to $D_{tsp}(P_n^*, i)$ by adjoining $2n-3, 2n-2$.
Therefore, $\{\{X \cup \{2n-1\}$ if X ends with $2n-2\} \cup \{X \cup \{2n\}$ if X ends with $2n-1\} \cup \{Y \cup \{2n-3, 2n-1\}\} \subseteq D_{tsp}(P_n^*, i)$. (9)

Conversely, suppose $Z \in D_{tsp}(P_n^*, i)$. Here all the elements of $D_{tsp}(P_n^*, i)$ ends with $2n-1$ or $2n$.

- If $2n-1 \in Z, 2n \notin Z$, then at least one vertex labeled $2n-2$ or $2n-3$ is in Z. *Suppose* $2n-3 \in Z, 2n-2 \notin Z$, then $Z = Y \cup \{2n-3, 2n-1\}$, for some $Y \in D_{tsp}(P_{n-2}^*, i-2)$. Suppose $2n-2 \in Z$, then $Z = X \cup \{2n-1\}$, for some $X \in D_{tsp}(P_n^* - \{2n\}, i-1)$.
- If $2n \in Z$, then $Z = X \cup \{2n\}$, for some $X \in D_{tsp}(P_n^* - \{2n\}, i-1)$.

Therefore, $D_{tsp}(P_n^*, i) \subseteq \{\{X \cup \{2n-1\}$ if X ends with $2n-2\} \cup \{X \cup \{2n\}$ if X ends with $2n-1\} \cup \{Y \cup \{2n-3, 2n-1\}\}$. (10)

From (9) and (10), we get $D_{tsp}(P_n^*, i) = \{\{X \cup \{2n-1\}$ if X ends with $2n-2\} \cup \{X \cup \{2n\}$ if X ends with $2n-1\} \cup \{Y \cup \{2n-3, 2n-1\}\}$, where $X \in D_{tsp}(P_n^* - \{2n\}, i-1)$ and $Y \in D_{tsp}(P_{n-2}^*, i-2)$.

Theorem 3.4. For every $n \geq 3, |D_{tsp}(P_n^*, i)| = |D_{tsp}(P_n^* - \{2n\}, i-1)| + D_{tsp}(P_{n-2}^*, i-2)|$.

Proof. It follows from Theorem 3.3. Here we state recursive formula for the twain secure perfect domination polynomial of $P_n^* - \{2n\}$.

Theorem 3.5. For every $n \geq 3$, $D_{tsp}(P_n^*, x) = x\left[D_{tsp}(P_n^* - \{2n\}, x) + xD_{tsp}(P_{n-2}^*, x)\right]$, with initial values, $D_{tsp}(P_1^*, x) = 2x + x^2, D_{tsp}(P_2^* - 4, x) = 2x^2 + x^3, D_{tsp}(P_2^*, x) = 2x^2 + 2x^3 + x^4, D_{tsp}(P_3^* - 6, x) = 2x^3 + 3x^4 + x^5$.

Proof. We have, $|D_{tsp}(P_n^*, i)| = |D_{tsp}(P_n^* - \{2n\}, i-1)| + D_{tsp}(P_{n-2}^*, i-2)|$. That is $d_{tsp}(P_n^*, i) = d_{tsp}(P_n^* - \{2n\}, i-1) + d_{tsp}(P_{n-2}^*, i-2)$. Therefore, $d_{tsp}(P_n^*, i)x^i = d_{tsp}(P_n^* - \{2n\}, i-1)\, x^i + d_{tsp}(P_{n-2}^*, i-2)x^i$. Which implies, $\sum d_{tsp}(P_n^*, i)x^i = \sum d_{tsp}(P_n^* - \{2n\}, i-1)\, x^i + \sum d_{tsp}(P_{n-2}^*, i-2)x^i$. That gives, $\sum d_{tsp}(P_n^*, i)x^i = x\sum d_{tsp}(P_n^* - \{2n\}, i-1)\, x^{i-1} + x^2 \sum d_{tsp}(P_{n-2}^*, i-2)x^{i-2}$. Which gives, $D_{tsp}(P_n^*, x) = x\left[D_{tsp}(P_n^* - \{2n\}, x) + xD_{tsp}(P_{n-2}^*, x)\right]$ with initial values, $D_{tsp}(P_1^*, x) = 2x + x^2, D_{tsp}(P_2^* - \{4\}, x) = 2x^2 + x^3, D_{tsp}(P_2^*, x) = 2x^2 + 2x^3 + x^4, D_{tsp}(P_3^* - \{6\}, x) = 2x^3 + 3x^4 + x^5$.

$n\backslash i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

$\boldsymbol{P_1^* - \{2\}}$	1														
$\boldsymbol{P_1^*}$	2	1													
$\boldsymbol{P_2^* - \{4\}}$	0	2	1												
$\boldsymbol{P_2^*}$	0	2	2	1											
$\boldsymbol{P_3^* - \{6\}}$	0	0	2	3	1										
$\boldsymbol{P_3^*}$	0	0	2	3	3	1									
$\boldsymbol{P_4^* - \{8\}}$	0	0	0	2	5	4	1								
$\boldsymbol{P_4^*}$	0	0	0	2	4	6	4	1							
$\boldsymbol{P_5^* - \{10\}}$	0	0	0	0	2	7	9	5	1						
$\boldsymbol{P_5^*}$	0	0	0	0	2	5	10	10	5	1					
$\boldsymbol{P_6^* - \{12\}}$	0	0	0	0	0	2	9	16	14	6	1				
$\boldsymbol{P_6^*}$	0	0	0	0	0	2	6	15	20	15	6	1			
$\boldsymbol{P_7^* - \{14\}}$	0	0	0	0	0	0	2	11	25	30	20	7	1		
$\boldsymbol{P_7^*}$	0	0	0	0	0	0	2	7	21	35	35	21	7	1	
$\boldsymbol{P_8^* - \{16\}}$	0	0	0	0	0	0	0	2	13	36	55	50	27	8	1

Table 1. $d_{tsp}(P_n^*, i)$ and $d_{tsp}(P_n^* - \{2n\}, i)$

4. Conclusion

This paper examines and analyzes centipede's twain secure perfect dominating sets and twain secure perfect domination polynomials using a recursive formula. We created the polynomial $D_{tsp}(P_n^*, x) = \sum_{i=n}^{2n} d_{tsp}(P_n^*, i)\, x^i$, which we term the twain secure perfect domination polynomial of P_n^*.

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