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Certified Domination Polynomials of Generalized Friendship Graphs

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Abstract: Let $G = (V, E)$ be a simple graph of order . The certified domination polynomial of G is the polynomial $D_{cer}(G, x) = \sum_{i=\gamma_{cer}(G)}^{|V(G)|} d_{cer}(G, i)\, x^i$, where $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating set of G and $d_{cer}(G, i)$ is the number of certified dominating sets of G of size i. Let n and $q \geq 3$ be any positive integer and $F_{q,n}$ be the generalized friendship graph formed by a collection of n cycles (all of order q), meeting at a common vertex. In this article, we study the certified domination polynomials of generalized friendship graphs $F_{3,n}$ $F_{4,n}$ and $F_{5,n}$

Key words: certified dominating set, certified domination number, certified domination polynomial.

Mathematics Subject Classification: 05C69,05C31,05

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $v \in V$, the open neighbourhood of v is the set $(v) = \{u \in V \,/\, uv \in E\}$ and the closed neighbourhood of v is the set $[v] = (v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v\in S}(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$.

The concept of dominating sets and the domination number in Graph Theory were first introduced by Oystein Ore and Claude Berge in 1960s. The concept of domination polynomial in Graph Theory was introduced by Saied Alikhani and Yee-hock Peng in 2009. The concept of certified domination in graphs was introduced by Dettlaf et al.,2020. They also further studied the concept in their subsequent work including its applications in real life situations. This motivaited us to study the certified domination polynomial of certain graphs.

Definition 1.2: A set $S \subseteq V$ is a **dominating set** of G, if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number (G) of G is the minimum cardinality of dominating set.

Definition 1.3: [6] A dominating set S is a **certified dominating set** of G if S has either zero or at least two neighbours in $V - S$. The certified domination number $\gamma_{cer}(G)$ of G is the minimum cardinality of certified dominating set.

Definition 1.4: Let G be a simple connected graph. Let $D_{cer}(G, i)$ be a family of all certified dominating sets of G with cardinality i and let $d_{cer}(G, i) = |D_{cer}(G, i)|$. Then the **certified domination polynomial** $D_{cer}(G, x)$ is defined as $D_{cer}(G, x) = \sum_{i=\gamma_{cer}(G)}^{|V(G)|} d_{cer}(G, i)$, where γ_{cer} (G) is the certified domination number of G.

Definition 1.5: [10] Generalized friendship graph denoted $F_{q,n}$ is collection of all n cycles (all of order q), meeting a common vertex. The following figures shows the examples of friendship graphs $F_{3,n}$ $F_{4,n}$ and $F_{5,n}$.

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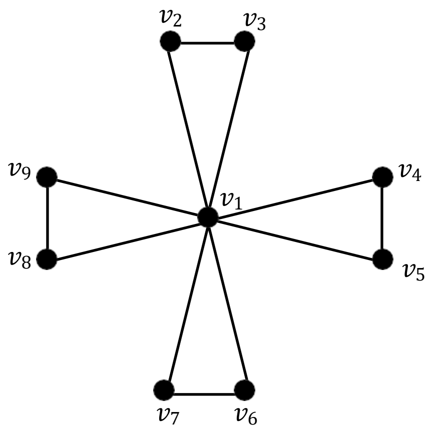

Figure 1 : Friendship graph $F_{3,4}$

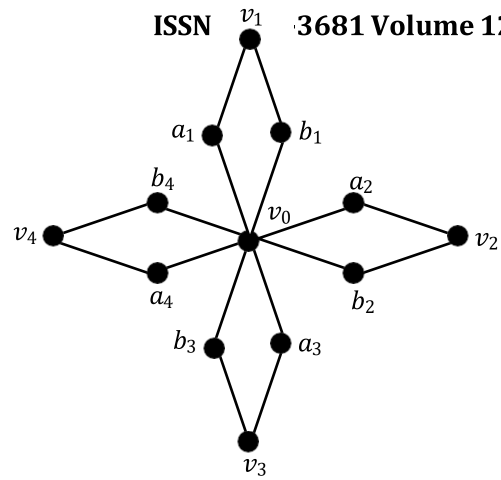

Figure 2 : Friendship graph $F_{4,4}$

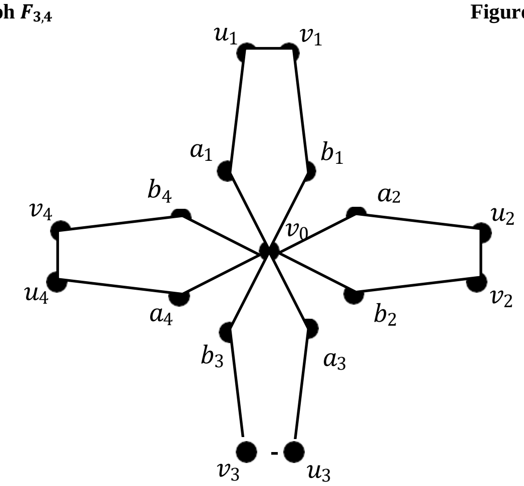

Figure 3 : Friendship graph $F_{5,4}$

Lemma 1.5: Let G be the graph with n vertices. Then

(i) $d_{cer}(G, n) = 1$

(ii) $d_{cer}(G, i) = 0$ if and only if $i < \gamma_{cer}(G)$ or $i = n - 1$ or $i > n$

(iii) $D_{\text{cer}}(G, x)$ has no constant term.

2. Certified domination polynomials of $F_{3,n}$, $F_{4,n}$ and $F_{5,n}$:

Let $D_{cer}(F_{q,n}, i)$ be the family of certified dominating sets of $F_{q,n}$ with cardinality i and let $d_{cer}(F_{q,n}, i) = |D_{\text{cer}}(F_{q,n}, i)|$ In this section, we shall investigate the certified dominating sets and certified domination polynomials of the friendship graphs $F_{3,n}$, $F_{4,n}$ and $F_{5,n}$

Theorem 2.1:

Let $F_{3,\text{n}}$ be a friendship graph with $2n + 1$ vertices. Then for all $n \in N$,

(i) $\gamma_{cer}(F_{3,n}) = 1$

$$(ii) d_{cer}(F_{3,n}, i) = \begin{cases} \binom{n}{\lfloor \frac{i}{2} \rfloor} & if\ i\ is\ odd\ but\ i \neq n \\ \binom{n}{\lfloor \frac{i}{2} \rfloor} + 2^n & if\ i = n\ and\ n\ is\ odd \\ 2^n & if\ i = n\ and\ n\ is\ even \\ 0 & otherwise \end{cases}$$

Proof:

Let $F_{3,n}$ be a friendship graph with $2n + 1$ vertices and $(F_{3,n}) = \{v_1, v_2, \ldots, v_{2n+1}\}$, where v_1 is the common vertex.

(i) Clearly the common vertex v_1 is the minimum certified dominating set. Hence $\gamma_{cer}(F_{3,n}) = 1$

(ii) For $n = 1$.

If $i = 1$, then $\{v_1\}$, $\{v_2\}$ and $\{v_3\}$ are the certified dominating sets, if $i = 2$, there are no certified dominating sets and if $i = 3$, then $\{v_1, v_2, v_3\}$ is the certified dominating set.

For all $n \geq 2$,

If i is odd and $i \neq n$, then the certified dominating set of $F_{3,n}$ is obtained by choosing $\lfloor \frac{i}{2} \rfloor$ copies of the cycle C_3 that joins with a common vertex from the n copies of the cycle C_3 that joins with a common vertex. Hence there are $\binom{n}{\lfloor \frac{i}{2} \rfloor}$ possible ways.

If $i = n$ and n is odd, then the certified dominating set of $F_{3,n}$ is obtained by choosing $\lfloor \frac{i}{2} \rfloor$ copies of the cycle C_3 that joins with a common vertex from the n copies of the cycle C_3 that joins with a common vertex. In addition to that select one vertex from each copies of cycle C_3 other than the common vertex. Hence there are $\binom{n}{\lfloor \frac{i}{2} \rfloor} + 2^n$ possible ways.

If $i = n$ and n is even, then the certified dominating set does not contain the common vertex v_1 and also it does not contain any two adjacent vertices. Therefore we need to select one vertex from each copies of cycle C_3 other than the common vertex v_1. Thus we get 2^n sets.

If $i > 2n + 1$, there are no certified dominating sets.

If i is even but $i \neq n$, there are no certified dominating sets.

Lemma 2.2:

The certified domination polynomial of the friendship graph $F_{3,n}$ with $2n + 1$ vertices is $(1 + x^2)^n + 2^n x^n$ for all $n \in N$.

Proof :

For $n = 1$, $D_{cer}(F_{3,1}, x) = 3x + x^3 = x(1 + x^2) + 2x$

For all $n \geq 2$,

If n is odd,

$$(F_{3,n}, x) = \sum_{i=1}^{2n+1} d_{cer}(F_{3,n}, i)\, x^i$$

$$= \binom{n}{0}x + \binom{n}{1}x^3 + \cdots + \binom{n}{\lfloor \frac{n-2}{2} \rfloor} x^{n-2} + \left\{\binom{n}{\lfloor \frac{n}{2} \rfloor} + 2^n\right\} x^n + \binom{n}{\lfloor \frac{n+2}{2} \rfloor} x^{n+2} + \cdots + \binom{n}{\lfloor \frac{2n+1}{2} \rfloor} x^{2n+1}$$ (from Theorem 2.1)

$= x\sum_{r=0}^{n}\binom{n}{r}x^{2r} + 2^n x^n$

$= (1 + x^2) + 2^n x^n$

If n is even,

$D_{cer}(F_{3,n}, x) = \sum_{i=1}^{2n+1} d_{cer}(F_{3,n}, i)\, x^i$

$= \binom{n}{0}x + \binom{n}{1}x^3 + \cdots + \binom{n}{\lfloor\frac{n-1}{2}\rfloor}x^{n-1} + 2^n x^n + \binom{n}{\lfloor\frac{n+1}{2}\rfloor}x^{n+1} + .. + \binom{n}{\lfloor\frac{2n+1}{2}\rfloor}x^{2n+1}$ (from Theorem 2.1)

$= x\sum_{r=0}^{n}\binom{n}{r}x^{2r} + 2^n x^n$

$= x(1 + x^2)^n + 2^n x^n$

Remark 2.3:

Sum of co-efficients of a certified domination polynomial of the friendship graph $F_{3,n}$ is 2^{n+1} for all $n \in N$.

i / $F_{3,n}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$F_{3,1}$	3	0	1														
$F_{3,2}$	1	4	2	0	1												
$F_{3,3}$	1	0	11	0	3	0	1										
$F_{3,4}$	1	0	4	16	6	0	4	0	1								
$F_{3,5}$	1	0	5	0	42	0	10	0	5	0	1						
$F_{3,6}$	1	0	6	0	15	20	20	0	15	0	6	0	1				
$F_{3,7}$	1	0	7	0	21	0	163	0	35	0	21	0	7	0	1		
$F_{3,8}$	1	0	8	0	28	0	56	256	70	0	56	0	28	0	8	0	1

Table 1: $(F3,n,i)$, the number of co-efficients of $F3,n$ with cardinality i

Theorem 2.4:

Let $F_{4,n}$ be a friendship graph with $3n + 1$ vertices. Then for all $n \in N$,

(i) $\gamma_{cer}(F_{4,n}) = n + 1$

$$(ii) d_{cer}(F_{4,n}, i) = \begin{cases} \binom{n}{\lfloor\frac{i}{2}\rfloor - \lceil\frac{n}{2}\rceil} & if\ i = n+1, n+3, \ldots, 3n+1\ but\ i \neq 2n \\ \binom{n}{\lfloor\frac{i}{2}\rfloor - \lceil\frac{n}{2}\rceil} + 1 & if\ i = 2n\ and\ n\ is\ odd \\ 1 & if\ i = 2n\ and\ n\ is\ even \\ 0 & otherwise \end{cases}$$

Proof :

Let $F_{4,n}$ be a friendship graph with $3n + 1$ vertices and $V(F_{4,n}) = \{v_0, v_j, a_j, b_j / 1 \le j \le n\}$, where v_0 is the common vertex, v_j represent the vertices on the cycle C_4 which are not adjacent to the common vertex v_0 , a_j and b_j represent the other vertices on the cycle which are adjacent to both the common vertex v_0 and v_j.

(i) Clearly $\{v_0, v_1, \ldots, v_n\}$ is the minimum certified dominating set. Hence $\gamma_{cer}(F_{4,n}) = n + 1$

(ii) For $n = 1$,

If $i = 2$, $\{v_0, v_1\}$ and $\{a_1, b_1\}$ are the certified dominating sets, if $i = 4$, clearly $d_{cer}(F_{4,1}, 4) = 1$ and if $i = 1,3$, there are no certified dominating sets.

For all $n \geq 2$,

If $i = n+1, n+3, \ldots, 3n+1$ but $i \neq 2n$, for the certified dominating set we need to select $\left\lfloor \frac{i}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil$ copies of cycle C_4 from the n copies of C_4 and one vertex from each of the remaining copies but that vertex is not adjacent to the common vertex. Hence there are $\binom{n}{\left\lfloor \frac{i}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil}$ possible ways.

If $i = 2n$ and n is odd, for the certified dominating set we need to select $\left\lfloor \frac{i}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil$ (that is $\left\lfloor \frac{n}{2} \right\rfloor$) copies of cycle C_4 from the n copies of C_4 and one vertex from each of the remaining copies but that vertex is not adjacent to the common vertex. In addition to that $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ is also a certified dominating set. Hence there are $\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} + 1$ possible ways.

If $i = 2n$ and n is even, then $\{a_j, b_j / 1 \leq j \leq n\}$ is a certified dominating set.

If $i < n+1$ and $i > 3n+1$, there are no certified dominating sets.

If $i = n+2, n+4, \ldots, 3n$ but $i \neq n$, there are no certified dominating sets.

Lemma 2.5:

The certified domination polynomial of the friendship graph $F_{4,n}$ with $3n+1$ vertices is $x^{n+1}(1+x^2)^n + x^{2n}$ for all $n \in N$.

Proof :

For $n = 1$, $D_{cer}(F_{4,1}, x) = 2x^2 + x^4 = x^2(1+x^2) + x^2$

For all $n \geq 2$,

If n is odd,

$D_{cer}(F_{4,n}, x) = \sum_{i=1}^{3n+1} d_{cer}(F_{4,n}, i)\, x^i$

$= \binom{n}{0}x^{n+1} + \binom{n}{1}x^{n+3} + \cdots + \binom{n}{\left\lfloor \frac{2(n-1)}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil} x^{2(n-1)} + \left\{\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} + 1\right\} x^{2n} + \binom{n}{\left\lfloor \frac{2(n+1)}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil} x^{2(n+1)} + \ldots + \binom{n}{n} x^{3n+1}$ (from Theorem 2.4)

$= x^{n+1} \sum_{r=0}^{n} \binom{n}{r} x^{2r} + x^{2n}$

$= x^{n+1}(1+x^2)^n + x^{2n}$

Similarly we prove for n is even

Remark 2.6:

Sum of co-efficients of a certified domination polynomial of the friendship graph $F_{4,n}$ is $2^n + 1$ for all $n \in N$.

i \ $F_{4,n}$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$F_{4,1}$	2	0	1																		
$F_{4,2}$	0	1	1	2	0	1															
$F_{4,3}$	0	0	1	0	4	0	3	0	1												
$F_{4,4}$	0	0	0	1	0	4	1	6	0	4	0	1									
$F_{4,5}$	0	0	0	0	1	0	5	0	11	0	10	0	5	0	1						
$F_{4,6}$	0	0	0	0	0	1	0	6	0	15	1	20	0	15	0	6	0	1			
$F_{4,7}$	0	0	0	0	0	0	1	0	7	0	21	0	36	0	35	0	21	0	7	0	1

Table 1: $d_{cer}(F_{4,n},i)$, the number of co-efficients of $F_{4,n}$ with cardinality i

Theorem 2.7:

Let $F_{5,n}$ be a friendship graph with $4n+1$ vertices. Then for all $n \in N$,

(i) $\gamma_{cer}(F_{5,n}) = n+1$

(ii) $$d_{cer}(F_{5,n}, i) = \begin{cases} \binom{n}{\lceil \frac{i}{3} \rceil - \lceil \frac{n+1}{3} \rceil} 2^{n-\lceil \frac{i}{3} \rceil + \lceil \frac{n+1}{3} \rceil} & if\ i = n+1, n+4, \dots, 4n+1\ but\ i \neq 2n \\ \binom{n}{\lceil \frac{i}{3} \rceil - \lceil \frac{n+1}{3} \rceil} 2^{n-\lceil \frac{i}{3} \rceil + \lceil \frac{n+1}{3} \rceil} + 3^n & if\ i = 2n\ and\ n \equiv 1 (mod 3) \\ 3^n & if\ i = 2n\ and\ n \not\equiv 1 (mod 3) \\ 0 & otherwise \end{cases}$$

Proof :

Let $F_{5,n}$ be a friendship graph with $4n+1$ vertices and $V(F_{5,n}) = \{v_0, u_j, v_j, a_j, b_j / 1 \le j \le n\}$, where v_0 is the common vertex, u_j and v_j represent the vertices on the cycle C_5 which are not adjacent to the common vertex v_0 , a_j represent the vertices on the cycle C_5 which are adjacent to both the common vertex v_0 and u_j and b_j represent the vertices on the cycle which are adjacent to both the common vertex v_0 and v_j.

(i) Clearly $\{v_0, u_1, u_2, \dots, u_n\}$ and $\{v_0, v_1, v_2, \dots, v_n\}$ is the minimum certified dominating sets. Hence $\gamma_{cer}(F_{5,n}) = n+1$

(ii) For $n = 1$,

If $i = 2$, $\{v_0, u_1\}, \{v_0, v_1\}, \{a_1, v_1\}$, $\{b_1, v_1\}$ and $\{a_1, b_1\}$ are the certified dominating sets, if $i = 5$, clearly $d_{cer}(F_{5,1}, 5) = 1$ and if $i = 1,3,4$, there are no certified dominating sets.

For all $n \geq 2$,

If $i = n+1, n+4, \dots 4n+1$ is odd but $i \neq 2n$, then the certified dominating set of $F_{5,n}$ is obtained by choosing $\lceil \frac{i}{3} \rceil - \lceil \frac{n+1}{3} \rceil$ copies of the cycle C_5 from the n copies and one vertex from each of the remaining copies of C_5 but that vertex is not adjacent to the common vertex v_0. Hence there are $\binom{n}{\lceil \frac{i}{3} \rceil - \lceil \frac{n+1}{3} \rceil} 2^{n-\lceil \frac{i}{3} \rceil + \lceil \frac{n+1}{3} \rceil}$ possible ways.

If $i = 2n$ and $n \equiv 1(mod 3)$, then the certified dominating set of $F_{5,n}$ is obtained by choosing $\lceil \frac{i}{3} \rceil - \lceil \frac{n+1}{3} \rceil$(that is $\lceil \frac{n}{3} \rceil$) copies of the cycle C_5 from the n copies and one vertex from each of the remaining copies of cycle C_5 but that vertex is not adjacent to the common vertex v_0. In addition to that select two vertices from each copies of cycle C_5 other than the common vertex and these vertices are not adjacent to each other. There are three sets of vertices satisfying this condition. Hence there are $\binom{n}{\lceil \frac{n}{3} \rceil} 2^{n-\lceil \frac{n}{3} \rceil} + 3^n$ possible ways.

If $i = 2n$ and $n \not\equiv 1(mod3)$, for the certified dominating set of $F_{5,n}$ we need to select two vertices from each copies of cycle C_5 other than the common vertex but these vertices are not adjacent to each other. Hence there are 3^n possible ways.

If $i < n + 1$, and $i > 4n + 1$, there are no certified dominating sets.

If $i = n + 2, n + 3, n + 5, n + 6, \ldots, 4n - 1, 4n$ (but $i \neq 2n$), there are no certified dominating sets.

Lemma 2.8:

The certified domination polynomial of the friendship graph $F_{5,n}$ with $4n + 1$ vertices is $x^{n+1}(2 + x^3)^n + (3x^2)^n$ for all $n \in N$.

Proof : For $n \equiv 1(mod3)$,

$(F_{5,n}, x) = \sum_{i=n+1}^{4n+1} d_{cer}(F_{5,n}, i)\, x^i$

$= \binom{n}{0}2^n x^{n+1} + \binom{n}{1}2^{n-1}x^{n+4} + \cdots + \left\{\binom{n}{n-\left\lfloor\frac{n}{3}\right\rfloor}2^{n-\left\lfloor\frac{n}{3}\right\rfloor} + 3^n\right\}x^{2n} + \cdots + \binom{n}{n}x^{4n+1}$ (from Theorem 2.8)

$= x^{n+1}\sum_{r=0}^{n}\binom{n}{r}2^{n-r}x^{3r} + 3^n x^{2n}$

$= x^{n+1}(2 + x^3)^n + (3x^2)^n$

Similarly we can prove for $n \not\equiv 1(mod3)$

Remark 2.9:

Sum of co-efficients of a certified dominating polynomial of the friendship graph $F_{5,n}$ is $2(3^n)$ for all $n \in N$.

i / $F_{5,n}$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$F_{5,1}$	5	0	0	1																
$F_{5,2}$	0	4	9	0	4	0	0	1												
$F_{5,3}$	0	0	8	0	27	12	0	0	6	0	0	1								
$F_{5,4}$	0	0	0	16	0	0	113	0	0	24	0	0	8	0	0	1				
$F_{5,5}$	0	0	0	0	32	0	0	80	243	0	80	0	0	40	0	0	10	0	0	1

Table 1: $d_{cer}(F_{5,n}, i)$, the number of co-efficients of $F_{5,n}$ with cardinality i

Conclusion:

In this paper, we have derived the important relation of $d_{cer}(F_{q,}, i)$ where $q = 3,4,5$. Using this relation we have to found out the certified domination polynomial of the friendship graphs $F_{3,}$, $F_{4,n}$ and $F_{5,n}$.

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