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# DOMINATION NUMBER OF DOUBLE GRAPH OF SOME GRAPHS

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**ABSTRACT**: A graph G consists of (V, E), indicates that V is its node set and E is its edge set. A subset D of vertices in a graph G is called a dominating set, then should each vertex in G be either included in D or is adjacent to at least one vertex in D. The minimum number of vertices required to form such a set is called the domination number of the graph, represented by $\gamma(G)$. In this work, we computed the domination number of double graph of Queen's graph, Lilly graph, Ladder graph, Ladder rung graph and Triangular belt graph.

**Keywords:** Dominating Set, Double graphs, Queen graph, Ladder graph, Lilly graph.

## 1. INTRODUCTION

A graph G consists of a combination of (V, E), where V represents the collection of nodes (or vertices) and E denotes the collection of edges connecting them. Graph theory principles are usually utilized in various fields to analysis and model completely different applications. This includes learning compounds, building bonds in chemistry and learning atoms. Domination rises in facility location problems (e.g., hospitals, fire stations), where the facilities are fixed and in order to get the closest facility a person needs to travel a minimum distance. Though new results in graph domination are being discovered and published, there are enormous number of open problems & conjectures in this field. Double graph was introduced in the paper by Emanuele Munarini et.al. in the year 2008. Domination in graphs was studied by Nawarat Ananchuen in the year 2010 [2] and in 2013 Preeti Gupta discussed about Domination in graphs with application. Domination and independent domination were studied by allan and laskar in 1978 [1] and Goddard and Henning in 2013 [4]. G. S. Domke et.al. discussed about Inverse domination number of a graph in 2004 and gave a result [3]. Fundamental of domination in graphs was studied by T.W. Haynes et.al. in 1998 [9].

**Definition 1:** Assuming that every node in the graph corresponds to a square on a chessboard and that every edge denotes a queen's permitted move, the queen graph is created.

**Definition 2:** The ladder graph $L_n$ is generated by performing the Cartesian product between the path graph $P_n$, containing n vertices, and the path graph $P_2$, which has 2 vertices. This operation is denoted as $L_n = P_n \times P_2$.

**Definition 3:** The triangular belt graph TB(n)(↓n), is formed by starting with the ladder graph $L_n$ and adding additional edges that connect each node $u_i$ to the node $v_{i+1}$ for every i from 1 to n.

**Definition 4:** The Lilly graph, denoted as $I_n$, for $n \geq 2$, is defined as the union of 2 star graphs $2K_{1,n}$, $n \geq 2$ and two path graphs $P_n$, $n \geq 2$. The construction of this graph involves connecting two-star graphs and two path graphs, all of which share a common central node.

## 2. MAIN RESULTS:

**Theorem 2.1:** Domination number of double graph of queen's graph $\gamma(D(Q_{2*n}))$ is always 2.

Proof:

Let $u_i$, $1 \leq i \leq 2n$ be the node set of queen's graph $Q_{2*n}$. The graph of the queen in m ×n chessboard is represented by $Q_{m*n}$, where each square represents a graph node. If a queen could move from one square to the other in a single move, then two squares are said to be adjacent if they are on the same row, column, or diagonal. If every square on the board is either a part of D or adjacent to at least one square in D, then the set D of squares is known as a dominating set of $Q_{m*n}$. Think about two copies of $Q_{2*n}$. Let $\{u_i, 1 \leq i \leq 2n\}$ be the node set of one copy and $\{v_i, 1 \leq i \leq 2n\}$ be the node set of another copy. Double graph of queen's graph $D(Q_{2*n})$ is obtained by adding edges $u_iv_j$ and $v_iu_j$ for every edge $u_iu_j$ of $Q_{2*n}$.

Each queen can control (dominates) all the squares in the same row that occupies and, at most, 3 squares in the opposite row. Therefore, to cover all the squares we place a queen in the 1st row and place another queen in the 2nd row which is adjacent to the first queen. So that 2 queens dominates all the vertices in the first copy and also all the vertices in the second copy. Let us consider D where the 2 vertices will be in D. The above set D is an MDS of $D(Q_{2*n})$ if, for each and every vertex u in D, removing u from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the vertex less than that of D cannot be a dominating set of $D(Q_{2*n})$. Therefore, D is a MDS with cardinality 2. (ref Fig 2.1)

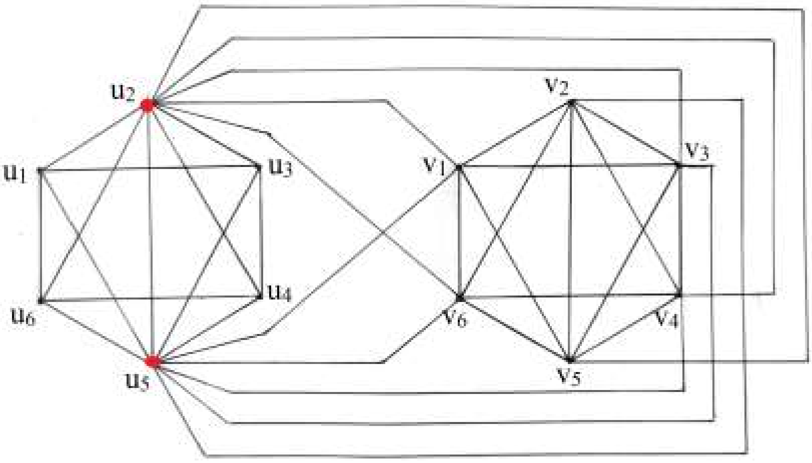

Fig 2.1 $\gamma(D(Q_{2*n})) = 2$

**Theorem 2.2**: Domination number of double graph of ladder graph $\gamma(D(L_n))$ is $2\left\lceil\frac{n}{3}\right\rceil$.

Proof:

Let $u_i$ and $u_i^{|}$ (for $1 \le i \le n$) be the nodes of the ladder graph $L_n$ . Now, consider creating two separate copies of $L_n$. The node set of the first copy is $\{u_i, u_i^{|}, 1 \le i \le n\}$, while the second copy has the node set $\{v_i, v_i^{|}, 1 \le i \le n\}$.

The double graph of ladder graph $D(L_n)$ is obtained by adding edges $u_i v_i^{|}$ and $u_i^{|} v_i$ for every edge $u_i u_i^{|}$ of $L_n$.

Partition the set of vertices of $D(L_n)$ into subsets each containing 3 vertices in each side of the ladder namely $\{\{u_1, u_1^{|}, u_2, u_2^{|}, u_3, u_3^{|}\}, \{u_4, u_4^{|}, u_5, u_5^{|}, u_6, u_6^{|}\}, \{u_7, u_7^{|}, u_8, u_8^{|}, u_9, u_9^{|}\}, \ldots\ldots. \{u_{n-2}, u_{n-2}^{|}, u_{n-1}, u_{n-1}^{|}, u_n, u_n^{|}\}\}$.

Case i) When $n$ *Congruent to* (0 *mod* 3)

Then, there are n/3 partition with exactly 6 vertices in all partition. Here the middle vertex in each side of the ladder in each partition are adjacent to the remaining nodes of the first copy and all the nodes in the corresponding components of the second copy of $L_n$. Let us consider $D=\{u_2, u_2^{|}, u_5, u_5^{|}, u_8, u_8^{|} \ldots\ldots.. u_{n-1}, u_{n-1}^{|}\}$. Since every node in $D(L_n)$ is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of $D(L_n)$ if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the node less than that of D cannot be a dominating set of $D(L_n)$ (ref Fig 2.2). Therefore, D is a MDS with cardinality $2\left\lceil \frac{n}{3} \right\rceil$.

Case ii) When $n$ *Congruent to* (1 *mod* 3)

Then, there are n/3 partition with exactly 6 vertices and one partition with two vertices $u_n$, $u_n^{|}$. Here the middle vertex in each side of the ladder in each partition are adjacent to the remaining vertices of the first copy and all the vertices in the corresponding components of the second copy of $L_n$ except $u_n$, $u_n^{|}$, and $v_n$, $v_n^{|}$ and the last vertex $u_n$, $u_n^{|}$ is adjacent to $v_n$, $v_n^{|}$. Let us consider $D= \{u_2, u_2^{|}, u_5, u_5^{|}, u_8, u_8^{|} \ldots\ldots.. u_n, u_n^{|}\}$. Since each node of $D(L_n)$ is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of $D(L_n)$ if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the node less than that of D cannot be a dominating set of $D(L_n)$. Therefore, D is a MDS with cardinality $2\left\lceil \frac{n}{3} \right\rceil$.

Case iii) When $n$ *Congruent to* (2 *mod* 3)

Then, there are n/3 partition with exactly 6 vertices and one partition with four vertices $u_{n-1}$, $u_{n-1}^{|}$, $u_n$, $u_n^{|}$. Here the middle vertex in each side of the ladder in each partition are adjacent to the remaining vertices of the first copy and all the vertices in the corresponding components of the second copy of $L_n$ except $u_{n-1}$, $u_{n-1}^{|}$ $u_n$, $u_n^{|}$, and $v_{n-1}$, $v_{n-1}^{|}$ $v_n$, $v_n^{|}$ and the last vertex $u_n$, $u_n^{|}$ is adjacent to $u_{n-1}$, $u_{n-1}^{|}$, $v_{n-1}$, $v_{n-1}^{|}$ $v_n$, $v_n^{|}$. Let us consider $D= \{u_2, u_2^{|}, u_5, u_5^{|}, u_8, u_8^{|} \ldots\ldots.. u_n, u_n^{|}\}$. Since each node of $D(L_n)$ is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of $D(L_n)$ if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the node less than that of D cannot be a dominating set of D($L_n$). Therefore, D is a MDS with cardinality $2\left\lceil\frac{n}{3}\right\rceil$.

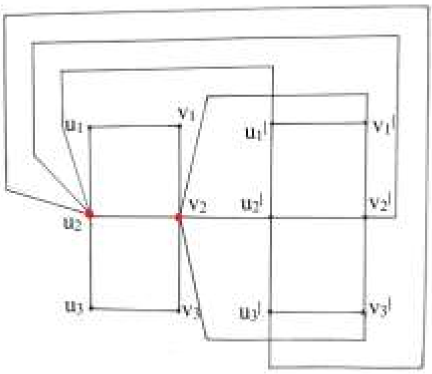

Fig 2.2: γ(D($L_3$)) = 2.

**Theorem 2.3:**

Domination number of double graph of triangular belt graph γ(D(TB(n)($\downarrow^n$))) is n - $\left\lfloor\frac{n}{3}\right\rfloor$.

Proof:

Let $u_i$ and $u_i^{|}$ (for $1 \le i \le n$) be the nodes of the Triangular belt graph. Now, consider creating two separate copies of TB(n)($\downarrow^n$). The node set of the first copy is $\{u_i, u_i^{|}, 1 \le i \le n\}$, while the second copy has the node set $\{v_i, v_i^{|}, 1 \le i \le n\}$.

The double graph of Triangular belt graph D(TB(n)($\downarrow^n$)) is constructed by adding edges $u_iv_i^{|}$ and $v_iu_i^{|}$ for every edge $u_iv_i$ of TB(n)($\downarrow^n$).

Partition the set of vertices of D(TB(n)($\downarrow^n$)) into subsets each containing 3 vertices in each side of the ladder namely $\{\{u_1, u_2, u_3, u_1^{|}, u_2^{|}, u_3^{|}\}, \{u_4, u_5, u_6, u_4^{|}, u_5^{|}, u_6^{|}\}, \{u_7, u_8, u_9, u_7^{|}, u_8^{|}, u_9^{|}\}, \ldots\ldots\{u_{n-2}, u_{n-1}, u_n, u_{n-2}^{|}, u_{n-1}^{|}, u_n^{|}\}\}$.

Case i) When $n\ Congruent\ to\ (0\ mod\ 3)$

Then, there are n/3 partition with exactly 6 vertices in all partition. Here the middle vertex in each side of the Triangular Belt Graph in each partition are adjacent to the remaining nodes of the first copy and all the nodes in the corresponding components of the second copy of TB(n)($\downarrow^n$). Let us consider D=$\{u_2, u_2^{|}, u_5, u_5^{|}, u_8, u_8^{|}\ldots\ldots u_{n-1}, u_{n-1}^{|}\}$. Since each node of D(TB(n)($\downarrow^n$)) is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of D(TB(n)($\downarrow^n$)) if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the vertex less than that of D cannot be a dominating set of D(TB(n)($\downarrow^n$)). Therefore, D is a MDS with cardinality n - $\left\lfloor\frac{n}{3}\right\rfloor$.

Case ii) When $n\ Congruent\ to\ (1\ mod\ 3)$

Then, there are n/3 partition with exactly 6 vertices and one partition with two vertices $u_n$, $u_n^{|}$. Here the middle vertex in each side of the Triangular belt graph in each partition are adjacent to the remaining nodes of the first copy and all the nodes in the corresponding components of the second copy of $TB(n)(\downarrow^n)$ except $u_n$, $u_n^{|}$, and $v_n$, $v_n^{|}$ and the last vertex $u_{n-1}$ is adjacent to $u_n$, $u_n^{|}$, $v_n$, $v_n^{|}$. Let us consider D={$u_2$, $u_2^{|}$, $u_5$, $u_5^{|}$, $u_8$, $u_8^{|}$…….. $u_{n-1}$}. Since each vertex of $D(TB(n)(\downarrow^n))$ is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of $D(TB(n)(\downarrow^n))$ if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the vertex less than that of D cannot be a dominating set of $D(TB(n)(\downarrow^n))$ (ref Fig 2.3). Therefore, D is a MDS with cardinality n - $\left\lfloor \frac{n}{3} \right\rfloor$.

Case iii) When $n$ *Congruent to* $(2 \bmod 3)$

Then, there are n/3 partition with exactly 6 vertices and one partition with four vertices $u_{n-1}$, $u_{n-1}^{|}$, $u_n$, $u_n^{|}$. Here the middle vertex in each side of the Triangular belt graph in each partition are adjacent to the remaining nodes of the first copy and all the nodes in the corresponding components of the second copy of $TB(n)(\downarrow^n)$ except $u_{n-1}$, $u_{n-1}^{|}$ $u_n$, $u_n^{|}$, and $v_{n-1}$, $v_{n-1}^{|}$ $v_n$, $v_n^{|}$ and the last vertex $u_n$, $u_n^{|}$ is adjacent to $u_{n-1}$, $u_{n-1}^{|}$, $v_{n-1}$, $v_{n-1}^{|}$ $v_n$, $v_n^{|}$. Let us consider D={$u_2$, $u_2^{|}$, $u_5$, $u_5^{|}$, $u_8$, $u_8^{|}$…….. $u_n$, $u_n^{|}$}. Since each vertex in $D(TB(n)(\downarrow^n))$ is either part of the set D or directly connected to a vertex in D, then the set D qualifies as a dominating set. The above set D is an MDS of $D(TB(n)(\downarrow^n))$ if, for each and every node u in D, removing that node from D results in a set that no longer dominates all the neighbours of u, i.e., any set containing the vertex less than that of D cannot be a dominating set of $D(TB(n)(\downarrow^n))$. Therefore, D is a MDS with cardinality n - $\left\lfloor \frac{n}{3} \right\rfloor$.

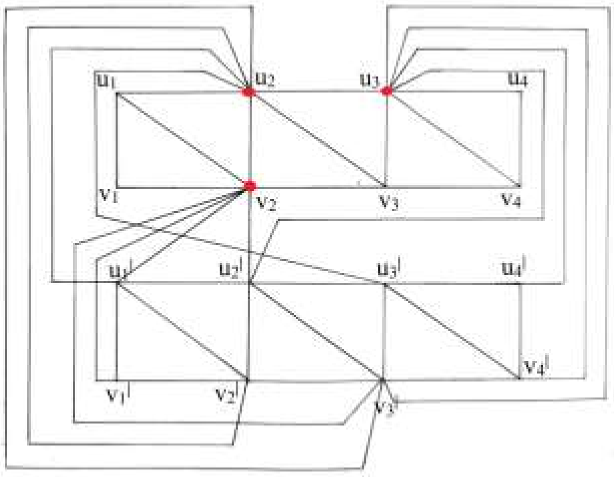

Fig 2.3: $\gamma(D(TB(4)(\downarrow^4))) = 3$

**Theorem 2.4:**

Theorem: Domination number of double graph of Lilly graph

$$\gamma(D(I_n)) = \begin{cases} n, & n \equiv 2,3 \pmod 4 \\ n+1, & n \equiv 0,1 \pmod 4 \end{cases}.$$

Proof:

Let $u_{i,}$ $1 \le i \le 2n-1$ be the nodes of $2P_n$, where $u_n$ is a common node and $\{v_i, 1 \le i \le n\} \cup \{w_i, 1 \le i \le n\}$ be the nodes of $2K_{1,n}$. Consider two copies of $I_n$. Let $\{u_{i,} 1 \le i \le 2n-1\}$ and $\{sv_i, 1 \le i \le n\} \cup \{w_i, 1 \le i \le n\}$ be the node set of one copy and $\{u_i^{|}, i \le 1 \le 2n-1\}$ and $\{v_i^{|}, 1 \le i \le n\} \cup \{w_i^{|}, 1 \le i \le n\}$ be the node set of another copy.

The double graph of lilly graph $I_n$ is constructed by adding the edges $u_iu_j^{|}$ and $u_ju_i^{|}$ for every edge $u_iu_j$ of $I_n$.

Separate the setoff nodes of $2P_n$ into subsets of 4 vertices each namely $\{\{u_1, u_2, u_3, u_4\}$ $\{u_5, u_6, u_7, u_8\}$ $\{u_9, u_{10}, u_{11}, u_{12}\}$ …………… $\{u_{n-3}, u_{n-2}, u_{n-1}, u_n\}\}$.

Case i) When $n\ Congruent\ to\ (0\ mod\ 4)$

Then there are $\frac{n}{4}$ partition with exactly 4 vertices in all the partition. Since the middle most 2 vertices in each partition are adjacent to all the preceding & Succeeding vertices of the component of $D(P_n)$ containing the partition, they dominate all the vertices of the component. Then, D is a MDS with cardinality is $\frac{n}{2}$.

Case ii) When $n\ Congruent\ to\ (1\ mod\ 4)$

There are $\frac{n}{4}$ partition and one more additional vertex Un. If we select middle most 2 vertices, we have $2[\left\lfloor\frac{n}{4}\right\rfloor - 1]$ are adjacent to all the preceding & Succeeding vertices of the component of $D(P_n)$ except $u_n$ & $v_n$. To cover $u_n$ & $v_n$ we include $u_{n-1}$ as the member of the dominating set. Then, D is a MDS with cardinality $\left\lfloor\frac{n}{2}\right\rfloor + 1$.

Case iii) When $n\ Congruent\ to\ (2\ mod\ 4)$

There are $\frac{n}{4}$ partition and exactly only two vertices in the last partition $u_{n-1}$, $u_n$. If we select middle most 2 vertices from each partition, then there is $2[\left\lfloor\frac{n}{4}\right\rfloor - 1]$ Vertices as the member of dominating set but two nodes $u_{n-1}$ & $u_n$ and corresponding vertices $v_{n-1}$ & $v_n$ of the last partition are not adjacent to any other vertices of the dominating set. So, without losing generality we select $u_{n-1}$ as the member of the dominating set. Then, D is a MDS with cardinality $\left\lfloor\frac{n}{2}\right\rfloor + 1$.

Case iv) When $n\ Congruent\ to\ (3\ mod\ 4)$

There are $\frac{n}{4}$ partition and exactly only three vertices in the last partition $u_{n-2}$, $u_{n-1}$, $u_n$. If we select middle most 2 vertices from each partition, then there is $2[\left\lfloor\frac{n}{4}\right\rfloor - 1]$ vertices as the member of dominating set but three vertices in the last partition $u_{n-2}$, $u_{n-1}$, $u_n$ and corresponding vertices $v_{n-2}$, $v_{n-1}$ & $v_n$ are not adjacent to any other vertices of the dominating set. So, without losing generality we select $u_{n-1}$, $u_n$ as the member of the dominating set (ref Fig 2.4). Then, D is a MDS with cardinality $\left\lfloor\frac{n}{2}\right\rfloor + 1$.

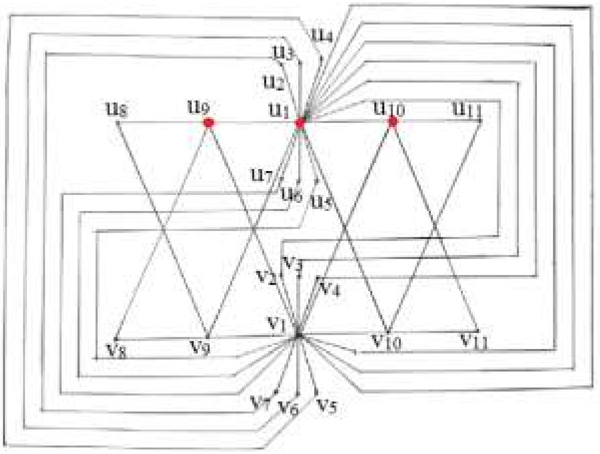

Fig 2.4: $\gamma(D(I_3)) = 3$

**Corollary 2.4:**

Domination number of double graph of Ladder rung graph is always 2n.

## 3. CONCLUSION:

Domination number of double graph of Queen’s graph, Ladder graph, Triangular belt graph and Lilly graph was discussed. And also, it follows that the domination number of the double graph formed from the ladder rung graph is consistently equal to 2n. In future research, domination of double graph and strong double graph of some other graphs can be analysed.

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