UtilitasMathematica

ISSN 0315-3681 Volume 122 (2), 2025

On Accurate Strong Non-Split Domination in Graphs

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Abstract. Accurate strong non-split dominating set in a graph with the relevant illustration is described in this manuscript. Minimal accurate strong non-split dominating set and accurate strong non-split domination numbers are specified for graphs. For some popular graphs, bounds on accurate strong non-split domination numbers are determined. Strong non-split domination number connections with other well-known dominating parameters are discovered. A few theorem relating to accurate strong non-split domination numbers are presented and obtained.

Keywords: Accurate non-split dominating set, Accurate strong non-split dominating set, Accurate non-split domination number, Accurate strong non-split domination number.

1 Introduction

Ore and Berge were the first to study dominating sets. V. R. Kulli et al brought the idea of Split domination[6], Non-split domination [7]. Strong split[8] and non-split domination[9] and Accurate dominating set[3]. K. Ameenal Bibi et al. introduced the notion of Accurate split and non-split domination[5]. In this paper, Accurate strong non-split dominating set and its domination numbers in graphs are stated and illustrated through examples.

2 Preliminaries

In this section, some basic definitions and theorems are given.

Definition 2.1: Let $G = (V, E)$ be a simple, finite connected and undirected graph. A set $D \subseteq V$ of a graph G is a dominating set(dom-set) if every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number(D-num) $\gamma(G)$ is the cardinality of a least dom-set of G.

Definition 2.2: A dom-set D of a graph $G = (V, E)$ is a non-split dom-set (NSD-set) if the induced subgraph $< V - D >$ is connected. The NSD-num $\gamma_{ns}(G)$ is the least cardinality of NSD-set.

Definition 2.3: A dom-set D of a graph $G = (V, E)$ is a strong NSD-set (SNSD-set) if the induced subgraph $< V - D >$ is complete. The $SNSD$-num $\gamma_{sns}(G)$ is the least cardinality of a SNSD-set.

Definition 2.4: A dom-set D of a graph $G = (V, E)$ is an accurate dom-set (AD-set) if V − D has no dom-set of cardinality |D|. The AD-num $\gamma_a(G)$ of G is the cardinality of a smallest AD-set of G.

Definition 2.5: A dom-set D of a graph $G = (V, E)$ is an accurate NSD-set ($ANSD$-set) if the induced sub graph $< V - D >$ is connected and it has no dom-set of cardinality $|D|$. The $ANSD$-num $\gamma_{ans}(G)$ of G is the least cardinality of an $ANSD$-set.

Theorem A[3]**:** For any complete graph, $\gamma_a(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1, n \geq 3$.

Theorem B[9]**:** For every graph, $\alpha_0(G) \leq \gamma_{sns}(G)$ where $\alpha_0(G)$ is the independence number.

3 Accurate strong non-split domination in graphs

Accurate strong non-split dom-set and accurate non-split dom-num of a graph are introduced in this section

Definition 3.1: An $ANSD$-set $D \subset V$ in a graph G is stated to be an accurate $SNSD$-set($ASNSD - set$), if the induced subgraph $< V - D >$ is complete. ASNSD sets that are not proper subsets of any other ASNSD sets are known as minimal ASNSD sets. A minimum $ASNSD$-set is a $ASNSD$-set of minimum cardinalities in a graph. The $ASNSD$-num $\gamma_{asns}(G)$ is the least cardinality of an $ASNSD$-set.

Example 3.2:

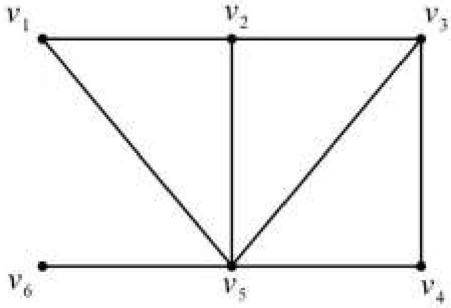

Fig 1. ASNSD-Graph

In this example, $\{v_5\}$ is a dom-set $\{v_5, v_6\}$ is a NSD −set. $\{v_3, v_4, v_6\}$ is a $SNSD$ −set. $\{v_1, v_2, v_3, v_4\}$ is an $ASNSD$ − set. Hence, $\gamma_{asns}(G) = 4$.

Observations:

1. For every connected graph, $n > 2,\ \gamma_{asns}(G) \leq n-1$.
2. For any Tree, Since, it is acyclic, $< V - D >$ is either K_1 or K_2

Bounds on $\boldsymbol{ASNSD}$ number for a few notable graphs. In this section, bounds on $ASNSD$ number for various graphs are determined.

Theorem 3.1: For every complete graph, $\gamma_{asns}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1, n \geq 3$.

Proof: By the theorem A, $\gamma_a(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. By definition, AD- set is said to be $ASNSD$-set if $< V - D >$ is complete. Therefore, $\gamma_{asns}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1, n \geq 3$.

Theorem 3.2: For any path graph $P_n, n \geq 3$,

$$\gamma_{asns}(P_n) = \begin{cases} n-2, & n \geq 5 \\ n-1, & n = 3,4 \end{cases}$$

Proof: Let us labelling the vertices of a path as $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$.
Case (i): If n=3,4, $\gamma_{asns}(P_n) = n-1$.
Case (ii): If $n \geq 5$, For finding accurate strong non split dominating set D in any path, choose alternative vertices $v_1, v_n, v_2, v_{n-1}, \ldots$ for D until to get $< V - D >$ is K_2.Therefore, $\gamma_{asns}(P_n) = n-2$.

Theorem 3.3: For any cycle graph $C_n, n \geq 3$,

$$\gamma_{asns}(C_n) = \begin{cases} n-2, & n \geq 5 \\ n-1, & n = 3,4 \end{cases}$$

Proof: Let us labelling the vertices of a cycle as $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$.
Case (i): If n=3,4, $\gamma_{asns}(C_n) = n-1$.
Case (ii): If $n \geq 5$, For finding accurate strong non split dominating set D in any cycle, choose alternative vertices $v_1, v_2, v_3, \ldots$ for D until to get $< V - D >$ is K_2.Therefore, $\gamma_{asns}(P_n) = n-2$.

Theorem 3.4: For any wheel graph $W_n, n \geq 4$,

$$\gamma_{asns}(W_n) = \begin{cases} n-1, & n = 4 \\ n-2, & n = 5,6 \\ n-3, & n \geq 7 \end{cases}$$

Proof: Let $'u'$ be the central vertex adjacent to all vertices $v_1, v_2, \ldots, v_{n-1}$ of a cycle. Then $\{u\}$ alone a non split dominating but not accurate and strong and also for any cycle C_{n-1},

$$\gamma_{asns}(C_{n-1}) = \begin{cases} n-3, & n-1 \geq 5 \\ n-2, & n-1 = 3,4 \end{cases}$$

Case (i): If n=4, then ASNSD-set contains $\{u\}$ and $n-2$ vertices. $\gamma_{asns}(W_n) = |\{u\}| + n - 2, \gamma_{asns}(W_n) = n-1$.

Case (ii): If n=5,6, then ASNSD-set contains $\{u\}$ and $n-3$ vertices. $\gamma_{asns}(W_n) = |\{u\}| + n - 3, \gamma_{asns}(W_n) = n - 2$.
Case (iii): If $n \geq 7$, ASNSD-set does not contains $\{u\}$ since to leave $< V - D >$ as complete graph. Therefore, ASNSD-set contains $n-3$ vertices. $\gamma_{asns}(W_n) = n - 2$.

Theorem 3.5: For any star graph S_n or $K_{1,n-1}, n \geq 3$

$$\gamma_{asns}(S_n) = n - 1$$

Proof: Let $'u'$ be the internal node or central node of the graph. If $D = V - \{u\}$, then $< V - D >$ is K_1. Therefore D is the $ASNSD$-set of S_n. Therefore, $\gamma_{asns}(S_n) = n - 1$.

Theorem 3.6: For any complete bipartite graph $K_{m,n}, 2 < m \leq n$,

$$\gamma_{asns}\left(K_{m,n}\right) = m + n - 2, \qquad m \leq n$$

Proof: Since each vertex in V_1 is adjacent to every vertex in V_2 and each vertex among V_1 and among V_2 are not adjacent, choosing vertices from V_1 and V_2 alternatively till K_2 is obtained for D-set gives ASNSD-set. Therefore, ASNSD-set contains $m-1$ vertices and $n-1$ vertices. Thus $\gamma_{asns}\left(K_{m,n}\right) = m + n - 2$.

Relation between $\boldsymbol{ASNSD}$ number and other parameters. In this section, relation between $ASNSD$ number and other parameter for complete and comb graph are determined.

Theorem 3.7: For every connected graph $G, \gamma_{sns}(G) \leq \gamma_{asns}(G)$.
Proof: Every $ASNSD$-set is a $SNSD$-set. Hence the result.

Theorem 3.8: For every connected graph G,

$$\gamma_{ans}(G) \leq \gamma_{asns}(G)$$

Proof: It is obviously true.

Theorem 3.9: For any spanning subgraph SH of $G, \gamma_{asns}(G) \leq \gamma_{asns}(SH)$.
Proof: Since vertex degree reduced in spanning subgraph and every ASNSD-set of SH is a ASNSD-set of G, Hence the theorem holds.

Corollary 3.1: If G is either complete or cycle graph, then $\gamma_{asns}(G) = \gamma_{asns}(SH) - 1$

Theorem 3.10: For every graph G, $\alpha_0(G) \leq \gamma_{asns}(G)$ where $\alpha_0(G)$ is the independence number of G.
Proof: By theorem 3.7 and theorem B
We have $\alpha_0(G) \leq \gamma_{sns}(G) \leq \gamma_{asns}(G)$.

Theorem 3.11: Let T_c be a comb graph, then (i) $for\ n \geq 5,\ \ \gamma_{asns}(T_c) = n - 2$ and (ii) $for\ n = 4,\ \gamma_{asns}(T_c) = \gamma_{sns}(T_c) + 1$
Proof:

(i) Let $S = \{e_1, e_2, \dots e_r\}$ be the end vertices and $P = \{p_1, p_2, \dots p_r\}$ be the cut vertices where $|V| = |E| + |P|$. Since each vertex is adjacent to an end vertex, non-split dominating set (NSD-set) has $S = \{e_1, e_2, \dots e_r\}$ vertices but not accurate and strong non-split dominating set. Then adding a vertex from the set P, we get $ANSD$-set and hence $< V - D >$ is a path. In path, only K_2 exists. Therefore, $\gamma_{asns}(T_c) = n - 2$.

(ii) It is obviously true.

Corollary 3.2: For $n \geq 2,\ \gamma_{ns}(T_c) = \frac{n}{2}$
Proof: Removing all $\frac{n}{2}$ end vertices leaves the graph connected and hence the result.

Corollary 3.3: Let T_c be a comb graph, then $for\ \ n \geq 5$,

$$\gamma_{asns}(T_c) = \gamma_{sns}(T_c)$$

Proof: For both $ASNSD$-set and $SNSD$-set, $< V - D >$ is complete. In T_c, $< V - D >$ will be K_2. Hence the result.

Corollary 3.4: Let T_c be a comb graph, then $for\ \ n \geq 5$,

$$\gamma_{asns}(T_c) = \gamma_{sns}(T_c) = 2(\gamma_{ns}(T_c) - 1)$$

Proof: By theorem 3.11,

$$\begin{aligned}\gamma_{asns}(T_c) = \gamma_{sns}(T_c) &= n - 2\\ &= 2\left(\frac{n}{2}\right) - 2\\ &= 2\gamma_{ns}(T_c) - 2.\end{aligned}$$

Note: ω is a clique number and ω-set is a clique set.

Theorem 3.12: For every graph G, which is not complete and ≥ 5 , $\gamma_{asns}(G) \leq n - \omega(G) + 2$, where is a clique number of G.
Proof: Let V_ω be the collection of vertices of ω-set then $\forall\ l \epsilon V_\omega$, $(V - V_\omega) \cup \{l\}$ is a $SNSD$-set but may be accurate or may not be accurate. Hence add any other vertex $m \epsilon V_\omega$ then $(V - V_\omega) \cup \{l, m\}$ is an $ASNSD$-set. Therefore, $\gamma_{asns}(G) \leq |(V - V_\omega) \cup \{l, m\}|$. Hence, $\gamma_{asns}(G) \leq n - \omega(G) + 2$.

Corollary 3.5: For any graph G, which is not complete with $\omega(G) \geq \delta(G)$ and ≥ 5 , $\gamma_{asns}(G) \leq n - \delta(G) + 2$, where ω is a clique number of G and $\delta(G)$ is a minimum degree of G.

Nordhaus –Gaddum(NG) type results. Lower and upper bounds on $ASNSD$-set for graph G and its complement graph $\bar{G}$ without any isolated vertices are determined in the succeeding theorems,. NG type results are obtained.

Theorem 3.13: For every graph G and $\bar{G}$ without any isolated vertices,

$$2 \leq \gamma_{asns}(G)$$

$$2 \leq \gamma_{asns}(\bar{G})$$

Proof: Let G and $\bar{G}$ be a graph without any isolated vertices. Then there is no universal vertices in both G and $\bar{G}$. Therefore, $\gamma_{asns}(G) > 1$ and $\gamma_{asns}(\bar{G}) > 1$. That is, $\gamma_{asns}(G) \geq 2$ and $\gamma_{asns}(\bar{G}) \geq 2$.

Theorem 3.14: For every graph G and $\bar{G}$ withoutany isolated vertices,

$$\gamma_{asns}(G) \leq (n-1)$$
$$\gamma_{asns}(\bar{G}) \leq (n-1)$$

Proof: By definition of $ASNSD$-set, $V - D$ has at least K_2 or K_1. Therefore, $\gamma_{asns}(G)$ and $\gamma_{asns}(\bar{G})$ has at most $(n-1)$ vertices.

Theorem 3.15: For any graph G and $\bar{G}$ with no isolated vertices,

$$4 \leq \gamma_{asns}(G) + \gamma_{asns}(\bar{G}) \leq 2(n-1)$$
$$4 \leq \gamma_{asns}(G).\gamma_{asns}(\bar{G}) \leq (n-1)^2$$

Proof: By theorem 3.13 and 3.14,

$$2 \leq \gamma_{asns}(G) \leq (n-1)$$
$$2 \leq \gamma_{asns}(\bar{G}) \leq (n-1)$$

This implies,

$$4 \leq \gamma_{asns}(G) + \gamma_{asns}(\bar{G}) \leq 2(n-1)$$
$$4 \leq \gamma_{asns}(G).\gamma_{ass}n(\bar{G}) \leq (n-1)^2$$

4 Conclusion

The concepts of $ASNSD$-set in graphs are addressed in this study. We find many limits on $ASNSD$ numbers. In the future, applications and connections between $ASNSD$ numbers with different well known domination parameters might be developed.

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