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ON ACCURATE STRONG SPLIT DOMINATION IN GRAPHS

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Abstract. *In this manuscript, accurate strong split dominating set, minimal accurate strong split dominating set and accurate strong split domination numbers in a graph are introduced. Bounds on accurate strong split domination numbers in graphs are obtained for some standard graphs. Relation between accurate strong split domination number and other well-known dominating parameters are generated.*

Keywords. *Dominating set, Strong split dominating set, Accurate strong split dominating set, Accurate strong split domination number.*

1 Introduction

The dominating set was first studied by Ore[1] and Berge[2]. Kulli wrote the book on theory of domination in graphs[3]. Kulli and Janakiram introduced the idea of split domination[7] and Strong split domination[8]. Kulli and Kattimani introduced the idea of accurate domination number[5]. Ameenal Bibi et al. in troduced the notion of accurate split domination[6]. In this manuscript, accurate strong split domination are introduced.

2 Preliminaries

A dominating set(dom-set) D of a graph $G = (V, E)$ is a subset of V such that every vertex not in D is adjacent to at least one vertex that is part of the dom-set. The number of vertices in the smallest dom-set of G is the domination number(dom-number) $\gamma(G)$ [4]. If the incited subgraph $< V - D >$ is disconnected, a graph's dom-set D is a split dom-set (or $SD - set$). The minimum cardinality of a $SD - set$ is $\gamma(G)$, is the split domination number[7]. A strong split dominating set ($SSD - set$) is a dom-set D of a graph $G = (V, E)$ if the inciteded subgraph $< V - D >$ is totally disconnected and there are at least two vertices. A $SSD -$

$set's$ minimum cardinality is the SSD number $\gamma_{ss}(G)$, refer [8] for SSD-number. If $V - D$ does not contain a dom-set of cardinality $|D|$, then the dom-set D of the graph $G = (V, E)$ is an accurate dom-set ($AD - set$). A smallest $AD - set$ of G is represented by the AD-number $\gamma_a(G)$ of G, refer [5] for AD-number. If the induced subgraph $< V - D >$ is disconnected and has no SD-set of cardinality $|D|$, the dom-set D of the graph $G = (V, E)$ is an accurate split dom-set (ASD-set). The minimum cardinality of an ASD-set is the ASD-number $\gamma_{as}(G)$ of G, refer [6] for ASD-number.

Definition 1. If $V - D$ contains no vertex cover of cardinality $|D|$, then a vertex cover D of G is an accurate vertex cover. The smallest cardinality of an accurate vertex cover of G is the accurate vertex covering number of G. [9]

Theorem 1. In any graph G,

(i) $\gamma(G) \leq \gamma_s(G)$ [7].

(ii) $\gamma_s(G) \leq \gamma_{ss}(G)$ [8].

(iii) $\gamma(G) \leq \gamma_a(G)$ [6].

(iv) $\gamma_a(G) \leq \gamma_{as}(G)$ [6].

(v) $\alpha_0(G) + P_0(G) = \gamma_{ss}(G)$. where $\alpha_0(G)$ and $P_0(G$ are the vertex covering number and number of isolates of a graph G respectively [8].

(vi) $\frac{n}{(\Delta(G)+1)} \leq \gamma(G)$ [5].

Theorem 2. In any cycle C_n with $n \geq 4$ vertices, $\gamma_{ss}(C_n) = [\frac{n}{2}]$ [8].

Theorem 3. In any wheel graph W_n with $n \geq 5, \gamma_{ss}(W_n) = [\frac{(n+1)}{2}]$ [8].

3 Accurate Strong Split Dominating set of a Graph

Definition 2. A dom-set $D \subset V$ in a graph G is purported to be an accurate strong split dom-set($ASSD - set$), if D is an AD-set and a SSD-set. An $ASSD$ set is a minimal $ASSD$-set if the proper subset of D not an $ASSD$-set. The $ASSD$- number $\gamma_{ass}(G)$ is the smallest cardinality of an $ASSD$-set.

Example 1. Let $G = (V, E)$ be a graph,

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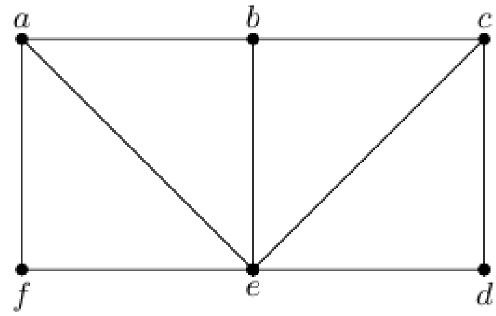

Fig. 1. Graph

$V = \{a, b, c, d, e, f\}$ and $E = \{(a, b), (a, e), (a, f), (b, c), (b, e), (c, d), (c, e), (d, e), (e, f)\}$. Here $\{e\}$ is a dom set. $\{c, e\}$ is a SD-set. $\{a, e, c\}$ is a SD-set. $\{a, e, c, b\}$ is an $ASSD$-sets. Hence, $\gamma_{ass}(G) = 4$.

Observations 3.1:

(i) Every Complete graph does not contains an $ASSD$-set. Since $< V - D >$ is not a totally disconnected graph.

(ii) If D is an $ASSD$-set, then $V - D$ has at least two vertices. Therefore $\gamma_{ass}(G) \leq n - 2$.

(iii) Proper subset of $ASSD$-set not an $ASSD$-set.

(iv) $\gamma(G) \leq \gamma_{ass}(G)$.

(v) Whole set V in any graph G not an $ASSD$-set.

(vi) For all graph $G, \gamma_{ass}(G) \leq n - \gamma_{ss}(G) + 2$.

In this paper, only finite simple connected but not complete graphs are considered.

3.1 Results for Some Standard Graphs

1. For any Path graph $P_n, n \geq 3$,

$$\gamma_{ass}(P_n) = \begin{cases} \frac{n}{2} + 1, & when\ n\ is\ even\ and\ n \neq 4 \\ 0, & when\ n = 4 \\ \frac{n-1}{2}, & when\ n\ is\ odd \end{cases}$$

2. For any Cycle Graph $C_n, n \geq 5$,

$$\gamma_{ass}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & when\ n\ is\ odd \\ \frac{n}{2} + 1, & when\ n\ is\ even \end{cases}$$

3. For any Wheel graph $W_n, n \geq 4$,

$$\gamma_{ass}(W_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & when\ n\ is\ odd \\ \frac{n}{2} + 1, & when\ n\ is\ even \end{cases}$$

4. For any Star graph S_n or $K_{1,n-1}, n \geq 3$

$$\gamma_{ass}(S_n) = 1$$

5. For any Complete Bipartite graph $K_{k,n}, 2 \leq k \leq n$ & $n \neq 2$,

$$\gamma_{ass}(K_{k,n}) = \begin{cases} k, & k < n \\ k + 1, & k = n \end{cases}$$

4 Relation between Accurate Strong Split Domination number with various parameters

Theorem 4. Every graph G which is connected,

$$\gamma(G) \leq \gamma_s(G) \leq \gamma_{ss}(G) \leq \gamma_{ass}(G)$$

Proof.

From theorem 1 (i), each SD-set in G is a dom-set in G. From theorem 1 (ii), every SSD-set in G is a SD-set of G. Now, since each $ASSD$-set in G is a SSD-set in G. Hence, we obtain (1).

Theorem 5. Every graph G which is connected,

$$\gamma(G) \leq \gamma_a(G) \leq \gamma_{as}(G) \leq \gamma_{ass}(G)$$

Proof. From theorem 1 (iii), each AD-set in G is a dom-set in G, From theorem 1 (iv), each ASD-set in G is an AD-set in G, Now, since each $ASSD$-set in G is an ASD-set in G, Therefore, we obtain (2).

Theorem 6. A dom-set of a graph G which is connected is an $ASSD$-set iff the conditions specified below are satisfied:

(i) $V - D$ has atleast two vertices.

(ii) For any two vertices $c, d \in V - D$, every $c - d$ path has at least one vertex of D.

(iii) $V - D$ has no dom-set of cardinality $|D|$.

Proof. Let D be an $ASSD$-set.

(i) By observation 3.1 (ii), $V -$ D has atleast two vertices.

(ii) Assume there exists an $c - d$ path containing no vertices of D. Then $< V - D >$ is connected. Hence every $c - d$ path has at least one vertex of D.

(iii) Suppose $V - D$ has a dom-set of cardinality $|D|$. Then D is not an AD-set. Hence Every $ASSD$-set satisfies the above three conditions.

Conversely, from (iii) D is an AD-set and from (i) and (ii) D is a SSD-set. Hence D is an $ASSD$-set.

Theorem 7. In any graph G, $\alpha_0(G) + P_0(G) \leq \gamma_{ass}(G)$. where $\alpha_0(G)$, $P_0(G)$ are the vertex covering number and the number of isolates of a graph G respectively.

Proof. From theorem 1(ii) and 4, we get the result.

Observation 3.2:

In any graph G, $\alpha_{a0}(G) + P_0 = \gamma_{ass}(G)$, $\alpha_{a0}(G)$ is the accurate vertex covering number.

Theorem 8. Let J be any spanning subgraph of a connected graph G, $\gamma_{ass}(J) \leq \gamma_{ass}(G)$

Proof. Since $\alpha_{a0}(J) \leq \alpha_{a0}(G)$ and J and G have no isolates, and by observation 3.2,

$$\gamma_{ass}(J) \leq \gamma_{ass}(G)$$

Theorem 9. In any graph G of order $n \geq 5$, $\gamma_{ass}(G) \leq n - \gamma(G) + 1$. Also, this bound is strict.

Proof. Let D be a smallest dom-set of G. Then for any $c \epsilon D$, $(V - D) \cup \{c\}$ is an $ASSD$-set of G. Thus $\gamma_{ass}(G) \leq |(V - D) \cup \{c\}| \leq n - \gamma(G) + 1$. Bound is sharp for all ladder graph.

Theorem 10. In any graph G, $\frac{n}{\Delta(G)+1} \leq \gamma_{ass}(G) \leq \frac{n\Delta(G)}{\Delta(G)+1} + 1$.

Proof. From theorem 1(iii) and 4, $\frac{n}{\Delta(G)+1} \leq \gamma(G) \leq \gamma_{ass}(G)$.

By the theorem 9, $\gamma_{ass}(G) \leq n - \gamma(G) + 1 \leq n - \frac{n}{\Delta(G)+1} + 1 \leq \frac{n\Delta(G)}{\Delta(G)+1}$

Hence the outcome.

Theorem 11. For any tree T with r articulation points(cut-vertices), $\gamma_{ass}(T) \leq r + 1$. Equality holds iff each articulation point is adjacent to an leaf vertex.

Proof. Let $a_1, a_2, \dots a_r$ be the r articulation points. Then for any end vertex $e \epsilon V$, $\{a_1, a_2, \dots a_r, e\}$ is an $ASSD$-set of T. Thus equation holds. Let $\gamma_{ass}(G) = r + 1$. Assume that articulation point exist that might not be adjacent to any leaf vertex, then $\{a_1, a_2, \dots a_r\}$ is an $ASSD$-set of T which is a contradiction. Thus each articulation point is adjacent to an leaf vertices. Converse is obvious.

Theorem 12. Assume that T be a tree with each articulation point is adjacent to an endvertex. Then

(i) $\gamma_{ass}(T) = \gamma_{ss}(T) + 1$

(ii) $\gamma_{ass}(T) = \gamma_{as}(T) = \gamma_a(T)$

Proof. (i) Let $a_1, a_2, \dots a_r$ be the articulation points. Then strong split dom set (SSD-set) has r vertices. Therefore, $\gamma_{ss}(T) = r$. Then by theorem 11, $\gamma_{ass}(G) = \gamma_{ss}(T) + 1$. (ii) D has r vertices and $V - D$ has r vertices. Therefore accurate dominating set has $r + 1$ vertices. Similarly, ASD-set and $ASSD$-set has $r + 1$ vertices. Thus, $\gamma_{ass}(T) = \gamma_{as}(T) = \gamma_a(T)$.

Theorem 13. Let T be any tree with $n \geq 6$, such that exactly two articula tion points adjacent to exactly two end vertices each, then $\gamma_{ass}(T) = \gamma_{ss}(T) = \left\lfloor\frac{n-4}{2}\right\rfloor + 1 = \left\lfloor\frac{r}{2}\right\rfloor + 1$, Where r is the number of articulation points.

Proof. Let a_1, a_2, a_3, a_4 are the end vertices and $a_5, a_6, \ldots, a_n$are the articulation point. Starting from a_5 for D as $a_5, a_n, a_7, a_{n-1}, \ldots$ until $< V - D >$ is totally disconnected.

Case (i): If n is even, D conatins $\frac{n-4}{2} + 1$ vertices. Case (ii): If n is odd, D contains $\left\lfloor\frac{n-4}{2}\right\rfloor + 1$ vertices. Therefore, SSD-set D has $\left\lfloor\frac{n-4}{2}\right\rfloor + 1$ vertices. Also, it is an $ASSD$-set since 4 pendant vertices are in $V - D$.

Theorem 14. In any cycle C_n with $n \geq 4$ vertices, $\gamma_{ass}(C_n) = n - \gamma_{ss}(C_n) + 1$

Proof. Since from theorem 2, $\gamma_{ss}(C_n) = \left\lceil\frac{n}{2}\right\rceil \Rightarrow n - \gamma_{ss}(C_n) + 1 = n - \left\lceil\frac{n}{2}\right\rceil + 1$

By result 2, When n is odd, $n - \gamma_{ss}(C_n) + 1 = n - \frac{(n+1)}{2} + 1 = \frac{n+1}{2} = \gamma_{ass}(C_n)$

When n is even, $n - \gamma_{ss}(C_n) + 1 = n - \frac{n}{2} + 1 = \frac{n}{2} + 1 = \gamma_{ass}(C_n)$

Theorem 15. For any wheel graph W_n, $\gamma_{ass}(W_n) = n - \gamma_{ss}(W_n) + 2$, when n is even. $\gamma_{ass}(W_n) = \gamma_{ss}(W_n)$, when n is odd

Proof. By result 3 and from theorem 3, When n is even, $\gamma_{ss}(W_n) = \left\lceil\frac{n+1}{2}\right\rceil = \frac{n+2}{2}$

$$n - \gamma_{ss}(W_n) + 2 = n - \frac{(n+2)}{2} + 2 = \frac{n+2}{2} = \frac{n}{2} + 1 = \gamma_{ass}(W_n)$$

When n is odd, $\gamma_{ss}(W_n) = \left\lceil\frac{n+1}{2}\right\rceil = \frac{n+1}{2}$

$$\gamma_{ss}(W_n) = \frac{n+1}{2} = \left\lceil\frac{n}{2}\right\rceil = \gamma_{ass}(W_n)$$

5 Norduaus–Gadduam Type Results

Theorem 16. Every graph G and $\bar{G}$ containing no isolated vertices,

(i) $2 \leq \gamma_{ass}(G) \leq (n-2)$

(ii) $2 \leq \gamma_{ass}(\bar{G}) \leq (n-2)$

Proof. Let G and $\bar{G}$ with no isolated vertices. Then there is no universal vertices in each G and $\bar{G}$. That is, $\gamma_{ass}(G) \geq 2$ and $\gamma_{ass}(\bar{G}) \geq 2$. By definition of $ASSD$-set, $V - D$ has at least 2 vertices. Therefore, $\gamma_{ass}(G)$ and $\gamma_{ass}(\bar{G})$ has at most $(n-2)$ vertices. $\gamma_{ass}(G) \leq (n-2)$ and $\gamma_{ass}(\bar{G}) \leq (n-2)$. Hence from (3) and (4), we obtain the result.

Theorem 17. Every graph G and $\bar{G}$ containing no isolated vertices, G and $\bar{G}$ with no isolated vertices,

(i) $4 \leq \gamma_{ass}(G) + \gamma_{ass}(\bar{G}) \leq 2(n-2)$

(ii) $4 \leq \gamma_{ass}(G). \gamma_{ass}(\bar{G}) \leq (n-2)^2$

Proof. By theorem 16, $2 \leq \gamma_{ass}(G) \leq (n-2)$ and $2 \leq \gamma_{ass}(\bar{G}) \leq (n-2)$. This implies the result.

6 Conclusion

In this manuscript, a variant of SSD is defined called $ASSD$ in graphs is defined. We found many bounds on $ASSD$ numbers. The exact values for some standard classes of graphs were found with Nordhaus-Gaddum results.

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