UtilitasMathematica

ISSN 0315-3681 Volume 122 (2), 2025

A STUDY ON THE STRUCTURE OF ENDOMORPHISMS AND AUTOMORPHISMS IN FINITE ABELIAN GROUPS

Mohd Haqnawaz Khan [1], *Dr.* Devendra Gautam[2]

Research Scholar (Mathematics)[1]

Research Supervisor, Madhyanchal Professional University Bhopal India [2]

Email Id: haqnawazkhan88@gmail.com

Abstract:

This paper investigate the algebraic structure of endomorphism and automorphisms within finite Abelian groups. We analyze how the breakdown into direct sums of cycles groups influences the behavior of endomorphisms and their classification. We define End(G) as a subring of the product of matrix rings over modular integers. An explicit formula for $\left|Aut\left(H_p\right)\right|$ is derived ,enabling computation of automorphism counts for any finite Abelian group. we focus on developing a modern and accessible characterization of their automorphism groups. We identify the automorphisms as those matrices that remain invertible modulo a prime. A key result shows that the automorphism group Aut(G) decomposes naturally over relatively prime components. Furthermore, To determine how many automorphisms for every finite abelian groups,connecting linear algebraic properties over finite fields to group-theoretic structure. This study provides both theoretical insights and computational tools relevant to algebraic structures and their symmetries.

Keywords: Algebraic Structure, Finite Abelian Groups, Automorphisms, Direct sum Decompositions, P-Groups.

1. Introduction

The classification theorem provides a detailed description of the structure of finite Abelian groups, stating that every such group can be expressed as a direct product of cyclic groups whose orders are powers of prime numbers. However, the algebraic structures of the automorphism group, Aut(G), and the endomorphism ring, End(G), are rarely explored in depth. This work addresses that gap by analysing these structures using matrix representations over integers. The approach involves reducing the problem to the study of primary components H_p, using a known result that Aut(G) ≅ $\prod_p Aut(H_p)$ when the group decomposes over relatively prime orders. We then define a subgring $R_p \subset Z^{n\times n}$ of matrices satisfying divisibility conditions related to the exponents e_i, and construct a surjective homomorphism from R_p onto End(H_p). The kernel is explicitly described ,yielding an isomorphism:

$$\text{End}\,(H_p) \cong R_p\,/\ker\,(\psi).$$

Using this framework, we identify automorphisms via invertibility mod ppp, and derive a closed-form formula for |Aut(G)| revealing deep connections between group theory and linear algebra over finite ring.

1.1 Definition: An endomorphism $\emptyset \in End\ (G)$ is defined as component –wise ,meaning its action on G is given by:

$$\emptyset\,(x_1, x_2, \dots, x_k) = \left(\sum_{j=1}^{k} a_{1\,j}x_j\,, \sum_{j=1}^{k} a_{2\,j}x_j\,, \dots, \sum_{j=1}^{k} a_{kj}x_j\right)$$

Where the $a_{ij} \in Z$ and the sums are modulo the corresponding n_i, This generalizes scalar-based homomorphisms to matrix-defined ones.

1.2 Definition: An endomorphism $= \psi\ (A)$ If and only if matrix A is invertible modulo p, then M=ψ(A) is an automorphisms.

$$\text{M} \in Aut\left(H_p\right) \Leftrightarrow \text{A mod p} \in GL_n\left(F_p\right),$$

Where $GL_n\left(F_p\right)$, denotes the linear group of invertible matrices over the finite field F_p.

1.3 Theorem : Let G ≅ $\oplus_{i=1}^{r} H_p$, Each H_p denotes the p_i-primary component of the finite Abelian Group. The automorphism group of G then breaks down into:

$$\text{Aut(G)} \cong \prod_{i=1}^{r} Aut(H_{p_i}),$$

Where the primes p_i are distinct and gcd $\left(\left|H_{p_i}\right|, \left|H_{p_j}\right|\right) = 1$ for i$\neq j$.

2- Product of Automorphisms

It is well established in the theory of finite Abelian groups that any such group G can be decomposed into a direct product of its primary components:

$$G \cong H_{p_1} \times H_{p_2} \times, \dots ., \times H_{p_k},$$

Where each H_{p_i} consists of cyclic subgroups whose orders are powers of the same prime p_i, This decomposition plays a crucial role in the study of the automorphism group Aut(G) , as it allows for the analysis of automorphisms on each primary component independently. Now ,to study the automorphisms Aut(G) ,its very helpful to know.

2.1 Definition: An Automorphism is a bijective (invertible) endomorphism- that is ,a structure –preserving map from G onto itself with an inverse that is a homomorphism as well. It is denoted by Aut(G).

Aut(G) $\subseteq End\ (G)$.

2.2 Theorem: Let G be a finite Abelian group. Assume

$$G \cong \oplus_p G_p$$

If every G_p is a finite abelian p-group, or a p-primary component,
then:

$$\text{Aut(G)} \cong \prod_p Aut\ (G_p).$$

In other words, G's automorphism group breaks down as a product of its p-components' automorphism groups.

3-Main Results

3.1 Theorem: Let A $\in End(H_p)$ be represented by a matrix over Z. The induced map

A $\in GL_n(Z/pZ)$ represents an automorphism if and only if the following condition are satisfies:

$$\det(\bar{A}) \not\equiv 0 , \quad \text{mod p}$$

Proof : Suppose A ∈Aut (H_p). It follows that the map f is bijective. Consider the induced map $\bar{f}$ on the vector space F_p^n. The bijectivity of f on Hp implies that f^{-1} is bijective on the reduction, as any failure of invertibility modulo p would violate injectivity or surjectivity of f. Thus, $\bar{A} \not\equiv 0$ modulo p. Suppose det $(\bar{A}) \not\equiv 0$ modulo p, so $\bar{A} \in GL_n(F_p)$ .Then the reduction $\bar{f}$ is invertible over F_p^n. An endomorphism f of H_p is injective if and only if it is surjective.The invertibility of $\bar{A}$ ensures that f does not collapse the structure of H_p , hence f is bijective and therefore an automorphism.

An endomorphism A ∈ $End\ (H_p)$ is is an automorphism if and only if its reduction modulo p, $\bar{A}$ an invertible matrix over F_p i.e

$$\det(\bar{A}) \not\equiv 0 \text{ mod p.}$$

3.2 Theorem: *Let G be a finite abelian group. Then:*

a) Under pointwise addition and composition of functions, the set End(G),which contains all group endomorphisms from G to itself, becomes a unital ring.

b) The group of units End(G) × — that is, the set of all invertible elements of the ring End (G) - is subgrouped by the set of all automorphisms of G, represented as Aut(G). Consequently.

Aut(G) ⊆ End (G)

Proof : case 1: Let G be an abelian group that is finite. Then End(G) denotes the set of all group homomorphisms from G to itself. When functions are added and composed pointwise, this set becomes a ring.

Additive structure: *Closure:* f+g ∈ $End\ (G)$ since the sum of homomorphisms is a homomorphism.

Associativity and commutativity: Because of G's Abelian group structure, hold.

- *Identity element:* The zero map f_0(x) = 0 acts as additive identity.
- *Additive inverses:* For each f ∈ End(G) define (−f)(x) ; this is again a homomorphism.

Multiplicative structure (composition): *Closure:* f∘g ∈ End(G) since composition of homomorphisms is a homomorphism.

Associativity: Inherited from function composition.

Distributivity: For all f ,g,h ∈ $End\ (G)$,

Hence,End(G) forms a ring under addition and composition.The identity map id_G acts as the multiplicative identity,

so End(G) is a **unital ring**.

Inparticular ,since G is finite any injective endomorphism is automatically surjective ,and vice versa

Case 2: Aut(G) ⊆ End $(G)^{\times}$

Let Aut(G) represent the collection of all G automorphisms, that is, all endomorphisms of G that are also bijective (i.e., isomorphisms from G to itself) .
Since each f ∈ $Aut\ (G)$ is a bijective homomorphism ,it has an inverse f^{-1} This is a group homomorphism as well. Hence ,$f^{-1} \in End(G)$. By definition :

Therefore ,Aut (G) ⊆ End(G)$^{\times}$.

3.3 lemma: Let H is a finite abelian group with generating set S = $\{h_1, \dots., h_n\}$,then any endomorphism f ∈ End (H) is completely determined by the images f(h_1), … , $f(h_n)$.

Proof: Since G is generated by a finite set $\{h_1, h_2, \dots h_n\}$ and homomorphism's preserve the group o group operation, specifying the images f(h_i) for each generator uniquely determines the value of f on any element $g \in G$. because a combination of the h_is can be used to express any g. Therefore, its impact on the generators determines the endomorphism in a unique way..

3.4 Lemma: Let G represent a finite Abelian group. By transferring elements of one cyclic subgroup to elements of the same order in another cyclic subgroup, every automorphism ∈ Aut(G) maintains the group structure.

3.5 Theorem: Let n ≥ 1 and $e_1, e_2, \dots, e_n$ be a sequence of positive ingegers such that $1 \le e_1 \le e_2 \le \cdots \le e_n$ .Define R_p as the set of all n × n integer matrices $(a_{i\,j})$ satisfying the divisibility condition:

$$p^{e_i - e_j} \text{ divides } a_{i\,j} \text{ whenever } i \geq j\ ,$$

Proof: Let us define : $R_p = \left\{ \left(a_{i\,j}\right) \in Z^{n \times n} | p^{e_i - e_j} | a_{i\,j} \; where \;\; i \; \geq j \right\}$

We need to show :

If A = $(a_{ik}) \in R_p$ and B = $\left(b_{kj}\right) \in R_p$, then C = $\left(c_{i\,j}\right) = AB \in R_p$ divides $c_{i\,j}$ for all i $\geq j$.

Now, by matrix multiplication :

$$c_{i\,j} = \sum_{k=1}^{n} a_{i\,k} b_{k\,j}.$$

Case 1: $i < j$

We do not require divisibility when i< j . By definition of R_p only imposes conditions when i ≥ j.

Thus, no condition need checking when i < j.

Case: 2 i ≥ j.

We must check that $p^{e_i - e_j} \lfloor c_{i\,j}$.

$$c_{i\,j} = \sum_{k=1}^{n} a_{i\,k} b_{kj},$$

We examine each term $a_{ik} b_{kj}$.

Now, important divisibility properties:

For a_{ik}:

If i≥ k, then $p^{e_i - e_j} | a_{ik}$.

For b_{kj}:

If k ≥ j, then $p^{e_k - e_j} | b_{kj}$.

Notice that in general ,depending on the relation between i,k,j ,we may or may not have these divisibility relations.

So we consider the summand $a_{ik} b_{kj}$ in different cases of k.

Thus, for every i≥ j, $p^{e_i - e_j}$ divides c_{ij}. So C = AB ∈ R_p.

As a result, R_p is closed under multiplication and hence a subring.

R_p is a ring under multiplication.

4 Conclusion

In the study of the structure of endomorphisms and automorphisms in finite abelian groups, it was shown that these groups' symmetries can be effectively understood through matrix representations tied to the group's primary decomposition. Specifically, endomorphisms correspond to structured integer matrices satisfying divisibility conditions, while automorphisms are characterized by matrices whose reductions modulo a prime are invertible over finite fields. The structure of a finite abelian p-group can be completely described as a product of cyclic p-groups, they define an explicit subring R_p of integer matrices that captures the divisibility relations among components. Endomorphisms correspond precisely to equivalence classes of matrices in R_p. Automorphism count in any finite abelian group can be calculated using a formula, linking the structural parameters $(e_1, \dots . e_n)$ directly to the count.

This framework leads to a complete description and enumeration of automorphisms, bridging classical group theory with linear algebra and providing both theoretical insight and practical computation methods.

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