Application of Triple ARA Integral Transforms to solving partial differential equations

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Abstract

In this paper, we present the development of a novel triple trans- form, termed the triple ARA transform. This transformative approach, specifically designed as a triple-integral transform, proves effective in addressing challenges posed by partial differential equations and vari- ous problem-solving scenarios. Our investigation delves into key char- acteristics of the triple ARA transform, emphasizing aspects such as its existence, linearity, and noteworthy findings pertaining to partial derivatives and the double convolution theorem. The versatility of the triple ARA transform is demonstrated through its application in solving a range of partial differential equations, including but not lim- ited to the homogeneous equation, and non-homogeneous equation in addressing diverse physical phenomena. Our study concludes by af- firming the user-friendly nature of our novel method in comparison to analogous transforms. The triple ARA transform emerges as a valu- able tool for researchers and practitioners seeking efficient solutions to a variety of mathematical and physical problems.

KEYWORDS: ARA integral transform; triple ARA; partial differen- tial equations.

INTRODUCTION

Integral transformations stand out as a highly effective approach for tackling the complexities of solving partial differential equations (PDEs). PDEs play a crucial role in mathematically representing a diverse array of phenomena in mathematical physics and various scientific domains, underscoring their inherent value [13, 14, 36, 26]. Leveraging integral transforms, these equa- tions can be systematically altered to unveil precise solutions, adding to the versatility of this methodology.

Numerous integral transforms have been developed and applied to ad- dress both partial and integral differential equations. These transformative techniques provide a means to derive exact solutions for target equations without the necessity of linearization or discretization. They play a cru- cial role in converting partial differential equations into ordinary equations through a single transform and into algebraic equations through a double integral transform. Notable examples encompass the Laplace transform [37], novel transform [10], M-transform [33], Sumudu transform [35,4,21], Elzaki transform [16], natural transform [19], Kamal transform [31], Aboodh trans- form [1], and ARA transform [27,11]. Additionally, various other transforms exist [22,29,34,6,7].

Double transformations have proven notably effective in handling partial differential equations (PDEs) when compared to alternative numeri- cal methods, particularly for solving PDEs involving unknown functions of two variables [2,13]. The literature has witnessed the development of exten- sions to double transforms, including the double Laplace transform, double Shehu transform [8], double Kamal transform [32], double Sumudu transform [17,15], double Elzaki transform [18], double Laplace-Sumudu transform [3], and ARA-Sumudu transform [28]. While all these double transforms can be viewed as specific instances of the general double transform described by Meddahi et al. [20], exploring special variants of double transforms remains valuable for comparative analysis. Such an exploration not only unveils the unique properties of each variant but also enhances our understanding of their optimal applications [24,23,30].

The intrinsic potency of transform techniques has spurred continu- ous research endeavours aimed at comprehending and refining their applications. Over time, numerous integral transforms have been conceptualized and applied to address both partial and integral differential equations. These transformative tools empower researchers and practitioners to derive exact solutions for target equations without resorting to linearization or discretiza- tion methods. Single transforms facilitate the conversion of partial differen- tial equations into ordinary equations, while triple integral transforms extend their utility, transforming them into algebraic equations [5, 9]. This adapt- ability underscores the significance of integral transforms in mathematical and scientific problem-solving.

Saadeh and others introduced the ARA transform [27], followed by Rania Saadeh's recent introduction of the double ARA transform [25], an innovative integral transform. In this paper, we present our development of a new transformative method known as the triple ARA transform, building upon the strengths of this potent transform approach.

The triple ARA transform is systematically detailed, elucidating its core properties and theorems. Our exploration includes the computation of triple ARA transform values for various elementary functions, alongside the establishment of novel relations between the triple ARA transform, partial derivatives, and the double convolution property. Demonstrating its practical utility, we apply these results to solve partial differential equations (PDEs), showcasing the transformative power of the triple ARA method.

Highlighting the distinct advantages of the triple ARA transform over its predecessor, the ARA transform, we underscore its simplicity and versa- tility in application. Notably, the triple ARA transform exhibits a unique trait: when applied to constants, the transformed constants remain constants devoid of variables in the results. This characteristic not only underscores the simplicity of the transform but also reduces computational complexity when employed to solve equations.

Amid the extensive research on solutions to partial differential equa- tions, the triple transform has emerged as a pivotal technique, renowned for its capacity to yield exact solutions rather than approximations. This study leverages the triple ARA transform specifically for solving PDEs, further substantiating its efficacy in producing precise solutions to complex mathe- matical problems.

This paper follows the structure: In section 2, we present fundamen- tal concepts and properties related to the ARA and double ARA transforms, In the section 3 we introduce the novel triple ARA transform along with associated properties and theorems, in section 4 we solve the homogeneous and nonhomogeneous third-order partial differential equation using ARA Trans- form, In the last conclusion section provides a summary of our results.

PRELIMINARY

In this section, we present the definition and fundamental properties of the ARA transform and double ARA transform.

Definition 2.1 (27)**.** The ARA integral transform of order m for a continuous function $r(t)$ defined on the interval $(0, \infty)$ is given by:

$$A_m[r(t)](s) = R(m, s) = s \int_0^\infty t^{m-1} e^{-st} r(t)\,dt, \; s > 0.$$

The ARA integral transform of order one, denoted as $A_1[r(t)]$, is defined as the following:

$$A_1[r(t)](s) = R(s) = s \int_0^\infty e^{-st} r(t)\,dt, \; s > 0$$

For simplicity, let us denote $A_1[r(t)] by A[r(t)]$. Our study specifically concentrates on the ARA transform of order one.

The inverse of ARA transform is

$$A^{-1}[R(s)] = \frac{1}{2\Pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{st}}{s} \, s \int_{0}^{\infty} e^{-st} r(t)\,dt$$

$$= \frac{1}{2\Pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{st}}{s} [R(s)]\,ds = r(t),\ t > 0.$$

Theorem 2.2. *If* r(t) *is a piecewise continuous integrable function in every finite interval* $0 < t \leq a$, $a > 0$, *and* r(t) *is of exponential order, that is, if it satisfies*

$$|r(t)| \leq Ke^{at}, \ t > \in [0, a],$$

Where K *is positive real constant independent of* t, *then the ARA transform* A *exists for all* $s > a$.

Proof. By employing the ARA transform definition, we get:

$$|A[r(t)]| = |R(s)| = s\int_0^\infty e^{-st} r(t)dt \ .$$

Using the property of improper integral, we get

$$|R(s)| = s\int_0^\infty e^{-st} r(t)dt \leq s\int_0^\infty e^{-st} r(t)dt \leq s\int_0^\infty e^{-st} |r(t)|dt \leq s\int_0^\infty e^{-st} Ke^{at}dt = sK\int_0^\infty e^{-(s-a)t}dt = \frac{sR}{s-\alpha}.$$

Hence, the improper integral converges for all $s > a$, and $A[q(t)]$ exists.
We state some basic properties of the ARA transform of order one
Assume that $R(s) = A[r(t)]$ and $Q(s) = A[q(t)]$ and $a, b \in R$. Then, we have

$$A[ar(t) + bq(t)] = aA_m[r(t)] + bA_m[q(t)].$$
$$A^{-1}[aR(m, s) + bQ(m, s)] = aA_m^{-1}[R(m, s)] + bA_m^{-1}[Q(m, s)]$$
$$A[t^a] = \frac{\Gamma(a + 1)}{s^{a+m-1}}, a > 0.$$
$$A\left[e^{at}\right] = \frac{s\Gamma(m)}{(s - a)^m}, a \in R.$$
$$A[\sin at] = \frac{as}{s^2 + a^2}, a \in R$$
$$A[r^m(t)] = s^m R(s) - \sum_{k=1}^{m} s^{m-k} r^{(k-1)}(0)$$

□

Definition 2.3. (**Double ARA transform**) Let $r(x, t)$ be continuous function of two positive variables x and t. Then double ARA of $r(x, t)$ is defined as [25+]

$$A_x A_t = R(g, h) = gh \int_0^\infty \int_0^\infty e^{-(gx+ht)} r(x, t)\,dx\,dt$$
$g, h > 0$, provided that the integral exists.

The inverse of double ARA transform is given by

$$A_x^{-1} A_t^{-1}[R(g, h)] = A_x^{-1} A_t^{-1}[R(g, h)] = \frac{1}{2\Pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{gx}}{g} dg \frac{1}{2\Pi i} \int_{b-i\infty}^{b+i\infty} \frac{e^{ht}}{h} R(g, h)\,dh = r(x, t).$$

TRIPLE ARA TRANSFORM

In this section, we introduce a triple ARA transform. We have discussed the fundamental properties and characteristic including the existence, unique- ness and the inverse of triple ARA transform.

Definition 3.1. Let $r(x, y, t)$ be a continuous function of three positive variables x, y, t then triple ARA of $r(x, y, t)$ is defined as.

$$A_x A_y A_t[r(x, y, t)] = R(g, h, i) = ghi \int_0^\infty \int_0^\infty \int_0^\infty e^{-(gx+hy+it)} r(x, y, t)\,dx\,dy\,dt.$$

$$g, h, i > 0$$

Provided that the integral exists.

Clearly, triple ARA transform is linear integral transform, as shown below

$$A_xA_yA_t[\mathring{A}r(x,y,t)+Bp(x,y,t)+Cq(x,y,t)]$$
$$=ghi\int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+t)}[\mathring{A}r(x,y,t)+Bp(x,y,t)+Cq(x,y,t)]dxdydz$$

$$=ghi\int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+t)}\mathring{A}(x,y,t)dxdydz$$
$$+ghi\int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+tit)}Bp(x,y,t)dxdydz$$
$$+ghi\int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+it)}Cq(x,y,t)dxdydz$$

$$=\mathring{A}A_xA_yA_t[r(x,y,t)]+BA_xA_yA_t[p(x,y,t)]+CA_xA_yA_t[q(x,y,t)]$$

Where $\mathring{A}$, B and C are constants and $A_xA_yA_t[r(x,y,t)]$, $A_xA_yA_t[p(x,y,t)]$ and $A_xA_yA_t[q(x,y,t)]$ are exists.

Definition 3.2. The inverse of the triple ARA transform is given by

$$A_x^{-1}A_y^{-1}A_t^{-1}[R(g,h,i)]=A_x^{-1}\ A_y^{-1}\ A_t^{-1}(R(g,h,i))\ \}$$
$$=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{gx}}{g}dg\ \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{e^{hy}}{h}dh\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{it}}{i}R(x,y,t)di$$
$$=r(x,y,t).$$

SOME PROPERTIES OF TRIPLE ARA TRANS- FORM

Triple ARA for some elementary functions:

i) let $r(x,y,t)=1$, $x>0$, $y>0$ and $t>0$ then

$$A_xA_yA_t[1]=ghi\int_0^\infty$$

$e^{-(gx+hy+it)}r(x, y, t)dxdydt \quad \int^{\infty} \int^{\infty} \int^{\infty}_{\infty} \quad \int^{\infty}$

0 0

$$= g \int_0^{\infty} e^{-gx}dx\, h \int_0^{\infty} e^{-hy}dy\, i \int_0^{\infty} e^{-it}dt$$

$= A_x[1]A_y[1]A_t[1] = 1$

where, $\mathrm{Re}(g) > 0$, $\mathrm{Re}(h) > 0$ and $\mathrm{Re}(i) > 0$

ii) let $r(x, y, t) = x^{\alpha}y^{\beta}t^{\gamma}$, $x > 0$, $y > 0$ and α, β, γ are constants

then $$A_xA_yA_t\left[x^{\alpha}y^{\beta}t^{\gamma}\right] = ghi\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(gx+hy+it)}\, x^{\alpha}y^{\beta}t^{\gamma}\, dxdydt$$

$$= g\int_0^{\infty} e^{-gx}x^{\alpha}dx\, h\int_0^{\infty} e^{-hy}y^{\beta}dy\, i\int_0^{\infty} e^{-it}t^{\gamma}dt$$

$$= A_x(x^{\alpha})A_y\left(y^{\beta}\right)A_t(t^{\gamma})$$

From the properties of ARA transform we obtain

$$A_xA_yA_t\left[x^{\alpha}y^{\beta}t^{\gamma}\right] = A_x(x^{\alpha})A_y\left(y^{\beta}\right)A_t(t^{\gamma})$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{g^{\alpha}h^{\beta}i^{\gamma}}$$

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(\gamma) > -1$

iii) let $r(x, y, t) = e^{gx+hy+it}$, $x, y, t > 0$ and α, β, γ are constants then

$$A_xA_yA_t\left[e^{gx+hy+it}\right] = ghi\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(gx+hy+it)}\, e^{\alpha x+\beta y+\gamma t}\, dxdydt$$

$$= A_x(e^{\alpha x})A_y\left(e^{\beta y}\right)A_t\left(e^{\gamma t}\right).$$

From the properties of ARA transform, we get

$$A_xA_yA_t\left[e^{gx+hy+it}\right] = \frac{ghi}{(g-\alpha)(h-\beta)(i-\gamma)}$$

$$A_x A_y A_t\, e^{i(gx+hy+it)} = \frac{ghi}{(g-i\alpha)(h-i\beta))(i-i\gamma)'}$$

using some properties of complex analysis, we find

$$A_x A_y A_t\, e^{i(gx+hy+it)} = \frac{ghi[ghi-\alpha\beta\gamma]+ighi[g\gamma+h\beta+i\alpha]}{(g^2-\alpha^2)(h^2-\beta^2)(i^2-\gamma^2)}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i},\ \cos x = \frac{e^{ix}+e^{-ix}}{2},\ \sinh x = \frac{e^{x}-e^{-x}}{2},\ \cosh x = \frac{e^{x}+e^{-x}}{2}$$

following we find TARA of the following functions as

$$A_x A_y A_t[\sin(\alpha x+\beta y+it)] = \frac{ghi(g\gamma+h\beta+\alpha i)}{(g^2+\alpha^2)(h^2+\beta^2)(i^2+\gamma^2)}$$

$$A_x A_y A_t[\cos(\alpha x+\beta y+it)] = \frac{ghi(ghi-\alpha\beta\gamma)}{(g^2+\alpha^2)(h^2+\beta^2)(i^2+\gamma^2)}$$

$$A_x A_y A_t[\sinh(\alpha x+\beta y+\gamma t)] = \frac{ghi(g\gamma+h\beta+i\gamma)}{(g^2-\alpha^2)(h^2-\beta^2)(i^2-\gamma^2)}$$

$$A_x A_y A_t[\cos h(\alpha x+\beta y+\gamma t)] = \frac{ghi(ghi-\alpha\beta\gamma)}{(g^2-\alpha^2)(h^2-\beta^2)(i^2-\gamma^2)}$$

iv) let $r(x,y,t) = J(c\sqrt{xyt})$ then

$$A_x A_y A_t\, J_0 c\sqrt{xyt} = ghi\int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+t)} J_0(\sqrt{xyt})\,dxdydt$$

$$= i\int_0^\infty e^{-it}dt.h\int_0^\infty e^{-hy}dy\ .\ g\int_0^\infty e^{-gx} J_0(c\sqrt{xyt})dx$$

$$= \frac{4ghi}{4ghi+c^2}$$

where J_0 is the modified bessel function of order zero

Existence condition of triple ARA transform:

Let $r(x,y,t)$ be a function of exponential orders α, β and γ as $x\to\infty, y\to\infty$ and $t\to\infty$ If $\exists$ a positive M such that $\forall x > X, y > Y\, and t > T$, we have

$|r(x,y,t)| \leq Me^{ax+\beta y+\gamma t}$
we can write $r(x,y,t) = 0 \ e^{ax+\beta y+rt}$ as

$$x \to \infty, y \to \infty \text{ and } t \to \infty, g > a, h > \beta \text{ and } i > \gamma$$

Theorem 4.1. . *Let* $r(x,y,t)$ *be a continuous on the region* $[0,X] \times [0,Y] \times [0,T]$ *of exponential orders* $a, \beta,$ *and* Y *then* $A_xA_yA_t[r(x,y,t)]$ *exist for* g, h *and provided that*
$\mathrm{Re}(g) > a, \mathrm{Re}(h) > \beta$ *and* $\mathrm{Re}(i) > \gamma$.

Proof. By using the definition of triple ARA transform, we find

$$|R(x,y,t)| = ghi \int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+it)} r(x,y,t)\,dxdydt$$

$$\leq ghi \int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+it)} |r(x,y,t)|\,dxdydt$$

$$\leq Mg \int_0^\infty e^{-(g-\alpha)x}dx \cdot h \int_0^\infty e^{-(h-\beta)s}dy \cdot i \int_0^\infty e^{-(-i-\gamma)t}dt$$

$$= \frac{Mghi}{(g-a)(h-\beta)(i-\gamma)} \cdot \mathrm{Re}(g) > a, \mathrm{Re}(h) > \beta \text{ and } \mathrm{Re}(i) > \gamma$$

Thus $A_xA_yA_t[r(x,y,t)]$ exist for g, h and i provided $\mathrm{Re}(g) > a$, $\mathrm{Re}(h) > \beta$ and $\mathrm{Re}(i) > \gamma$ □

Some theorems and properties of triple ARA trans- form:

Theorem 3.3: $A_xA_yA_t[r(x,y,t)] = A_xA_yA_t[w(x)v(y)u(t)]$

$$= ghi \int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+it)}[w(x)v(y)u(t)]\,dxdydt$$

$$= g\int_0^\infty w(x)e^{-g(x)}dx \cdot h\int_0^\infty v(y)e^{-h(y)}dy \cdot i\int_0^\infty u(t)e^{-i(t)}dt.$$

$$= A_x[w(x)]A_y[v(y)]A_t[u(t)]$$

Properties
Property 1 (Shifting Property):

Let $r(x, y, t)$ be a continuous function and $A_{xyt}[r(x, y, t)] = R(g, h, i)$ then

$$A_x A_y A_t \; e^{ax+\beta y+\gamma t} r(x, y, t) = \frac{ghi}{(g-a)(h-\beta)(i-\gamma)} \cdot R(g-a, h-\beta, i-\gamma)$$

Proof. : From definition of triple ARA transform, we find

$$A_x A_y A_t \; e^{\alpha x+\beta y+\gamma t} r(x, y, t) = \frac{ghi}{(g-\alpha)(h-\beta)(i-\gamma)} \cdot R(g-\alpha, h-\beta, i-\gamma)$$

$$A_x A_y A_t \; e^{ax+\beta y+\gamma t} r(x, y, t) = ghi \int_0^\infty \int_0^\infty \int_0^\infty e^{-(g-a)x-(h-\beta)y-(-i-\gamma)t} r(x, y, t)\,dxdydt$$

$$= \frac{ghi}{(g-a)(h-\beta)(i-\gamma)} (g-a, h-\beta, i-\gamma) \int_0^\infty \int_0^\infty \int_0^\infty e^{-(g-a)x-(h-\beta)y-(i-\gamma)t} r(x, y, t)\,dxdydt$$

$$= \frac{g}{g-\alpha} \cdot \frac{h}{h-\beta} \cdot \frac{i}{i-\gamma} \cdot R(g-a, h-\beta, i-\gamma)$$

□

1.1 Property 2 (Periodic Property):

Let $A_x A_y A_t[r(x, y, t)]$ exist, where $r(x, y, t)$ describes a periodic function of periods a, and y such that,

$$r(x + a, y + \beta, t + y) = r(x, y, t) \; \forall x, y, t$$

then,

$$R(g, h, i) = \frac{1}{1 - e^{-(ga+h\beta+iv)}} \, ghi \int_0^a \int_0^\beta \int_0^y e^{-(gx+hy+t)} r(x, y, t)\,dxdydt$$

Proof. : By employing the definition of triple ARA transform, we get:

$$A_x A_y A_t[r(x, y, t)] = ghi \int_0^\infty \int_0^\infty \int_0^\infty e^{-(gx+hy+t)} r(x, y, t)\,dxdydt..$$

By employing the properties of improper integral, the above equation can be expressed as follows:

$$A_x A_y A_t[r(x, y, t)] = ghi \int_0^{d}\int_0^{\beta}\int_0^{y} e^{-(gx+hy+t)} r(x, y, t)\, dx\, dy\, dt + ghi \int_a^{\infty}\int_\beta^{\infty}\int_y^{\infty} e^{-(gx+hy+t)} r(x, y, t)\, dx\, dy\, dt$$

.

Putting $x = a + \theta$, $y = \theta + \tau$ and $t = y + \xi$. On the second integral in above equation, we get

$$R(g, h, i) = ghi\, i \int_0^{\infty}\int_0^{\infty}\int_0^{\beta} e^{-(gx+hy+t)}\, r(x, y, t)\, dx\, dy\, dt$$
$$+ ghi \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(g(a+\rho)+h(\beta+\tau)+i(\gamma+\xi))} \times r(\alpha + \rho, \theta + \tau, \gamma + \xi)\, d\rho\, d\tau\, d\xi$$

By employing the periodicity of the function $r(x, y, t)$ above equation can be expressed as follows:

$$R(g, h, i) = ghi \int\int_0^{C}\int_0^{\beta}\int_0^{y} e^{-(gx+hy+t)}[r(x, y, t)]dxdydt$$
$$+e^{-(ga+h\beta+ii)} ghi \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(g\rho+ht+i\xi)[r(\rho,\tau,\xi)]d\rho d\tau d\xi}$$

By employing the definition of triple ARA transform, we get

$$R(g, h, i) = ghi \int_0^{a}\int_0^{\beta}\int_0^{y} e^{-(gx+hy+t)}[r(x, y, t)]dxdydt + e^{-(ga+h\beta+iv)} R(g, h, i)$$

Thus, equation can be simplified as,

$$R(g, h, i) = \frac{1}{1 - e^{-(ga+h\beta+i)}} ghi \int_0^{a} \int_0^{\beta} \int_0^{y} e^{-(gx+hy+ti)r} r(x, y, t)dxdydt$$

□

1.2 Property 3 (Heaviside Function):

Let $A_xA_yA_t[r(x, y, t)]$ exist and $A_xA_yA_t[r(x, y, t)] = R(g, h, i)$
Then,

$$A_xA_yA_t [x - \delta_1, y - \delta_2, t - \delta_3]\, H\, [x - \delta_1, y - \delta_2, t - \delta_3] = e^{-g\delta_1 - h\delta_2 - i\delta_3} R(g, h, i)$$

Where $H [x - \delta_1, y - \delta_2, t - \delta_3]$ is the Heaviside unit step function defined as,

$$H [x - \delta_1, y - \delta_2, t - \delta_3] = \begin{cases} 1 & x > \delta_1, y > \delta_2, t > \delta_3 \\ 0 & \text{otherwise} \end{cases}$$

By employing the definition of triple ARA transform, we get

$$A_xA_yA_t [r (x - \delta_1, y - \delta_2, -\delta_3)]\, H\, [x - \delta_1, y - \delta_2, t - \delta_3]$$
$$= ghi \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-(gx+hy+tit)} [r (x - \delta_1, y - \delta_2, t - \delta_3)]\, H\, [x - \delta_1, y - \delta_2, t - \delta_3]\, dxdydt$$
$$= ghi \int\int\int_0^{\infty} e^{-(gx+hy+tit)} [r (x - \delta_1, y - \delta_2, t - \delta_3)]\, dxdydt$$

Putting $x - \delta_1 = \rho, y - \delta_2 = \tau$ and $t - \delta_3 = \xi$ in above equation we get

$$A_x A_y A_t [r (x - \delta_1, y - \delta_2 t - \delta_3)\, H\, (x - \delta_1, y - \delta_2, t - \delta_3)]$$
$$ghi \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-g(\delta_1+\rho)-h(\delta_2+\tau)-(\delta_3-\xi)} r(\rho, t, \xi)d\rho d\tau d\xi$$

Thus, above equation can be simplified as.

$$A_xA_yA_t\,[r(x-\delta_1, y-\delta_2, t-\delta_3)]\,H\,[x-\delta_1, y-\delta_2, t-\delta_3]$$
$$= e^{-g\delta_1-h\delta_2-i\delta_3}\,ghi\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-g\rho-ht-i\xi} r(\rho, \tau, \zeta)\,d\rho\,dt\,d\xi$$
$$= e^{-g\delta_1-h\delta_2-i\delta_3} R(g, h, i).$$

Theorem 5.1. *Convolution theorem:*
Let $A_xA_yA_t[r(x, y, t)]$ *and* $A_xA_yA_t\,[r_1(x, y, t)]$ *exist and*

$$A_xA_yA_t[r(x, y, t)] = R(g, h, i),\quad A_xA_yA_{tt}\,[r_1(x, y, t)] = R_1(g, h, i)$$

Then

$$A_xA_yA_t\,[r(x, y, t) *** r_1(x, y, t)] = \frac{1}{ghi} R(g, h, i)R_1(g, h, i)$$

Where

$$r(x, y, t) *** r_1(x, y, t) = \int_0^{x}\int_0^{y}\int_0^{t} r(x-\rho, y-\tau, t-\xi)\,r_1(\rho, \tau, \xi)\,d\rho\,d\tau\,d\xi$$

*And the symbol *** denotes the triple convolution w.r.t.* x, y *and* t

Proof. By using the definition of triple ARA transform we get

$$A_xA_yA_t\,[r(x, y, t) *** r_1(x, y, t)] = ghi$$
$$= ghi\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(gxx+hy+it)}\,dx\,dy\,dt \int_0^{x}\int_0^{y}\int_0^{t} r(x-h-\rho, y-t, t-\xi)\,r(t)$$
$$\times\; r_1(\rho, \tau, \xi)\,d\rho\,dt\,d\xi]$$

By employing the Heaviside unit step function, above equation can be expressed as follows:

$$A_xA_yA_t[r(x, y, t))^{***}r_1(x, y, t)]$$
$$= \int_0^x\int_0^y\int_0^t r_1(\rho, \tau, \xi)\, d\rho\, d\tau\, d\xi$$
$$\times\ ih \int_0^\infty\int_0^\infty\int_0^\infty e^{-g\rho-h\tau-i\xi}\ e^{-gx-hy-it}\ r(x-\rho, y-\tau, t-\xi)\, H(x-\rho, y-\tau, t-\xi)\, dx\, dy\, dt$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty r_1(\rho, \tau, \xi) d\rho d\tau d\xi\ \ e^{-g\rho-h\tau-i\xi} R(g, h, i)$$
$$= R(g, h, i) \int_0^\infty\int_0^\infty\int_0^\infty e^{-g\rho-h\tau-i\xi} r_1(\rho, \tau, \xi) d\rho d\tau d\xi$$
$$A_xA_yA_t\,[r(x, y, t) * * * r_1(x, y, t)] = \frac{1}{ghi} R(g, h, i) R_1(g, h, i).$$

□

1.3 Property 4 (Derivative Properties):

Let $r(x, y, t)$ be a continuous function and $A_xA_yA_t[r(x, y, t)] = R(g, h, i) then$, we obtain the following derivative properties:

1.
$$A_xA_yA_t\left[\frac{\partial r(x, y, t)}{\partial t}\right] = iR(g, h, i) - iA_xA_y r(x, y, 0)$$

2.
$$A_xA_yA_t\left[\frac{\partial r(x, y, t)}{\partial y}\right] = hR(g, h, i) - hA_xA_t r(x, 0, t)$$

3.
$$A_xA_yA_t\left[\frac{\partial r(x, y, t)}{\partial x}\right] = gR(g, h, i) - gA_yA_t r(0, y, t)$$

4.
$$A_xA_yA_t\left[\frac{\partial^2 r(x, y, t)}{\partial t^2}\right] = ghi \int_0^\infty\int_0^\infty\int_0^\infty e^{-(gx+hy+t)} \frac{\partial^2 r(x, y, t)}{\partial t^2}\, dxdydt$$

..

$$= gh \int_0^\infty\int_0^\infty e^{-(gx+hy)} dxdy\, i \int_0^\infty e^{-it} \frac{\partial^2 r(x, y, t)}{\partial t^2} dt$$

Using integration by parts for above integration, we get

$$= gh \int_0^\infty \int_0^\infty e^{(qg+hv)} dxdy \int_0^\infty e^{-q\frac{\partial r(x,y,t)}{\partial t}} \int_0^\infty e^{-h\frac{\partial r(x,y,t)}{\partial t}} dt$$

$$= gh \int_0^\infty \int_0^\infty e^{(qg+hv)} dxdy \int_0^\infty \frac{\partial r(x,y,0)}{\partial t} — ir(x,y,0) + i^2 \; e^{ir(x,y,t)} dt$$

$$A_x A_y A_t \frac{\partial^3 r(x,y,t)}{\partial x \partial y \partial t} = i^2 R(g,h,i) - i^2 A_x A_y [r(x,y,0)] - iA_x \frac{\partial r(x,0,0)}{\partial t} \ldots (5)$$

Similarly, we can prove with respect to *x* and *y*:

5) $$A_x A_y A_t \frac{\partial^2 r(x,y,t)}{\partial x \partial y} = h^2 R(g,h,i) - h^2 A_x [r(x,0,t)] - hA_x \frac{\partial r(x,0,t)}{\partial y} \ldots (6)$$

6) $$A_x A_y A_t \frac{\partial^2 r(x,y,t)}{\partial x \partial t} = g^2 R(g,h,i) - g^2 A_y [r(0,y,t)] - gA_y \frac{\partial r(0,y,t)}{\partial x} \ldots (7)$$

7) $$A_x A_y A_t \frac{\partial^3 r(x,y,t)}{\partial x \partial y \partial t} = ghi \int_0^\infty \int_0^\infty \int_0^\infty e^{-(gx+hy+it)} \frac{\partial^3 r(x,y,t)}{\partial x \partial y \partial t} dx\, dy\, dt$$

$$\int_0^\infty \int_0^\infty = ghi \int_0^\infty e^{-(gx+it)} dx\, dt \int_0^\infty e^{-hy} \frac{\partial^3 r(x,y,t)}{\partial t \partial x \partial y} dy$$

Using integration by parts for above integration, we get:

$$= ghi \int_0^\infty \int_0^\infty e^{-(gx+it)} dx\, dt \quad -hy \frac{\partial^2 r}{\partial t \partial x} \Big|_0^\infty + h \int_0^\infty e^{-hy} \frac{\partial^2 r}{\partial t \partial x} dy$$

$$= ghi \int_0^\infty \int_0^\infty e^{-(gx+it)} dx\, dt \; - \frac{\partial^2 r(x, 0, t)}{\partial t \partial x} + ghz^{i} \int_0^\infty \int_0^\infty e^{-(gx+hy+it)} \frac{\partial^2 r}{\partial t \partial x} dx\, dy\, dt$$

$$= -ghi \int_0^\infty e^{-it} dt \int_0^\infty e^{-gx} \frac{\partial^2 r(x, 0, t)}{\partial t \partial x} dx + ghz^{i} \int_0^\infty \int_0^\infty e^{-(gx+it)} dy\, dt \int_0^\infty e^{-gx} \frac{\partial^2 r}{\partial t \partial x} dx$$

$$= -ghi \int_0^\infty e^{-it} dt \; e^{-gx} \frac{\partial r(x, 0, t)}{\partial t} \Big|_0^\infty + g \int_0^\infty e^{-gx} \frac{\partial r(x, 0, t)}{\partial t} dx$$

$$+ghz^{i} \int_0^\infty \int_0^\infty e^{-(hy+it)} dy\, dt \; e^{-gx} \frac{\partial r}{\partial t} \Big|_0^\infty + g \int_0^\infty e^{-gx} \frac{\partial r}{\partial t} dx$$

$$= ghi \; e^{-it} r(0, 0, t) \Big|_0^\infty + \int_0^\infty \int_0^\infty e^{-it} r(0, 0, t) dt \; - g^2 hi \int_0^\infty e^{-gx} dx \int_0^\infty e^{-it} \frac{\partial r(x, 0, t)}{\partial t} dt$$

$$-ghz^{i} \int_0^\infty e^{-hy} dy \int_0^\infty e^{-it} \frac{\partial r(0, y, t)}{\partial t} dt + g^2 h^2 i \int_0^\infty \int_0^\infty e^{-(gx+hy)} dx\, dy \int_0^\infty e^{-it} \frac{\partial r}{\partial t} dt$$

$$= -ghi r(0, 0, 0) + ghi A_t r(0, 0, t) + g^2 hi \int_0^\infty e^{-gx} r(x, 0, 0) dx - g^2 hi \int_0^\infty \int_0^\infty e^{-(gx+it)} r(x, 0, t) dx\, dt$$

$$+ghz^{i} \int_0^\infty e^{-hy} r(0, y, 0) dy - ghz^{i} \int_0^\infty \int_0^\infty e^{-(hy+it)} r(0, y, t) dy\, dt$$

$$- g^2 h^2 i^2 \int_0^\infty \int_0^\infty \int_0^\infty e^{-(gx+hy+it)} r(x, y, t) dx\, dy\, dt$$

Solution of the homogeneous equation:

In this part, we solve the homogeneous third-order equation using ARA Transform.

Example 1. Consider the non-homogeneous third-order differential equation.

$$\frac{\partial^3 f(x, y, t)}{\partial x \partial y \partial t} - f(x, y, t) = 0 \ldots (9)$$

With initial and boundary conditions:

$$f(x, y, 0) = e^{x+y}, \quad f(x, 0, t) = e^{x+t}, \quad f(0, y, t) = e^{y+t}, \quad f(0, 0, 0) = 1$$

$$f(x, 0, 0) = e^{x}, \quad f(0, y, 0) = e^{y}, \quad f(0, 0, t) = e^{t}$$

When we apply double ARA transform on initial conditions, we get:

$$A_x A_y [e^{x+y}] = \frac{g}{g-1} \frac{h}{h-1} \frac{i}{i-1}$$

$$A_x A_t [e^{x+t}] = \frac{g}{g-1} \frac{h}{h-1} \frac{i}{i-1}$$

When we apply ARA transform on boundary conditions, we have:

$$A_x(e^{x}) = \frac{g}{g-1}, \quad A_y(e^{y}) = \frac{h}{h-1}, \quad A_t(e^{t}) = \frac{i}{i-1}$$

By applying a triple ARA transform on equation (9) and using the above condition, we obtain:

$$(ghi - 1)A_xA_yA_t[f(x, y, t)] = \frac{g^2h^2i^2 - ghi}{(g-1)(h-1)(i-1)}$$

$$A_xA_yA_t[f(x, y, t)] = \frac{ghi(ghi-1)}{(g-1)(h-1)(i-1)} \times \frac{1}{(ghi-1)}$$

$$A_xA_yA_t[f(x, y, t)] = \frac{ghi}{(g-1)(h-1)(i-1)}$$

By taking the triple ARA inverse on both sides, we get:

$$f(x, y, t) = e^{x}e^{y}e^{t}$$

Solution of the nonhomogeneous equation:

In this part, we solve the nonhomogeneous third-order equation using ARA Transform.

Example 2. Consider the third-order Mbocatra partial differential equation.

$$\frac{\partial^3 f(x, y, t)}{\partial x \partial y \partial t} + f(x, y, t) = 3e^{-x^2yt} \quad (10)$$

With initial and boundary conditions:

$$f(x, y, 0) = e^{-x^2y}, \quad f(x, 0, t) = e^{-xt}, \quad f(0, y, t) = e^{-2yt}, \quad f(0, 0, 0) = 1$$

$$f(x, 0, 0) = e^{x}, \quad f(0, y, 0) = e^{y}, \quad f(0, 0, t) = e^{t}$$

When we apply double ARA transform on initial conditions we obtain:

$$A_x A_y [e^{-x^2y}] = \frac{g}{g+1} \frac{h}{h+2}, \quad A_x A_t [e^{-xt}] = \frac{h}{h+2} \frac{i}{i-1}$$

$$A_y A_t [e^{-2yt}] = \frac{g}{g+1} \frac{i}{i-1}$$

When we apply ARA transform on boundary conditions, we have:

$$A_x (e^{x}) = \frac{g}{g+1}, \quad A_y (e^{y}) = \frac{h}{h+2}, \quad A_t (e^{t}) = \frac{i}{i-1}$$

Applying triple ARA transform on equation (10), we get:

$$-\text{ghi}\,[f(0,0,0)] + \text{ghi}\,A_t[f(0,0,t)] + \text{ghi}\,A_y[f(x,0,0)] + \text{ghi}\,A_x[f(0,y,0)]$$
$$-\text{ghi}\,A_xA_t[f(x,0,t)] - \text{ghi}\,A_yA_t[f(0,y,t)] - \text{ghi}\,A_xA_y[f(x,y,0)] + \text{ghi}\,A_xA_yA_t[f(x,y,t)]$$
$$+A_xA_yA_t[f(x,y,t)] = 3\frac{g}{g+1}\cdot\frac{h}{h+2}\cdot\frac{i}{i-1}$$

$$(\text{ghi}+1)A_xA_yA_t[f(x,y,t)] =$$

$$= \frac{3\text{ghi}}{(g+1)(h+2)(i-1)} + \text{ghi}\frac{gh}{(g+1)(h+2)} + \text{ghi}\frac{gi}{(g+1)(i-1)}$$
$$+\text{ghi}\frac{hi}{(h+2)(i-1)} - \text{ghi}\frac{h}{h+2} - \text{ghi}\frac{i}{i-1} + \text{ghi} - \text{ghi}\frac{g}{g+1}$$

$$(\text{ghi}+1)A_xA_yA_t[f(x,y,t)] = \frac{3\text{ghi} - 2\text{ghi} + g^2h^2i^2}{(g+1)(h+2)(i-1)}$$

$$A_xA_yA_t[f(x,y,t)] = \frac{\text{ghi}(\text{ghi}+1)}{(g+1)(h+2)(i-1)}\cdot\frac{1}{(\text{ghi}+1)}$$

$$A_xA_yA_t[f(x,y,t)] = \frac{\text{ghi}}{(g+1)(h+2)(i-1)}$$

Taking triple inverse ARA transform on both sides, we get:

$$f(x,y,t) = e^{-x^2yt}$$

Conclusion

The triple ARA transform stands out as a promising and powerful tool in the realm of integral transforms, specifically tailored to address challenges posed by partial differential equations. Through our comprehensive investigation, we have elucidated key characteristics of the transform, establishing its ex- istence, and linearity, and unveiling noteworthy findings related to partial derivatives and the double convolution theorem. The versatility of the triple ARA transform has been effectively demonstrated through its successful application in solving a spectrum of partial differential equations, ranging from fundamental equations like the Homogeneous to the Non-homogeneous equation, all of which are integral in understanding diverse physical phenomena. Our study has not only showcased the efficacy of the triple ARA transform but has also emphasized its transformative capabilities by providing accurate solutions to model integral equations, thereby yielding exact results. This transformative approach not only enhances our understanding of mathematical and physical problems but also offers a user-friendly alternative compared to analogous transforms.

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